Journal of Quantitative Spectroscopy & Radiative Transfer 230 (2019) 86–105

Contents lists available at ScienceDirect

Journal of Quantitative Spectroscopy & Radiative Transfer

journal homepage: www.elsevier.com/locate/jqsrt

Review

Electromagnetic scattering by discrete random media. II: The coherent field

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ARTICLE INFO

Article history:
Received 22 January 2019
Revised 14 March 2019
Accepted 18 March 2019
Available online 19 March 2019

ABSTRACT

The computation of the coherent field in the case of a plane electromagnetic wave obliquely incident on a discrete random layer with non-scattering boundaries is addressed. For dense media, the analysis is based on a special-form solution for the conditional configuration-averaged exciting field coefficients, and is restricted to the computation of the so-called zeroth-order fields without a special treatment of the boundary regions. In this setting, we calculate the coherent fields reflected and transmitted by the layer, and the coherent field inside the layer. We found that these fields are analytically equivalent to plane electromagnetic waves, and investigated the fulfillment of the boundary conditions for the electric fields at the layer interfaces. The results are then particularized to the cases of normal incidence and a semi-infinite discrete random medium. For sparsely distributed particles, we present a self-consistent derivation of the coherent field and discuss the Twersky and Foldy approximations.

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https://doi.org/10.1016/j.jqsrt.2019.03.011

1. Introduction

In the first part of this series [1], we derived the dispersion equation and gave a prescription for computing the configuration-averaged exciting field in the case of a plane electromagnetic wave obliquely incident on a discrete random layer with non-scattering boundaries. Using these results as a starting point we continue our survey with the analysis of the coherent field inside and outside the scattering medium. To compute these fields in the case of dense media, we will use the techniques developed by Waterman and Truell [2], Fikioris and Waterman [3,4], and Tsang and Kong [5,6] for calculating the coherent field in a semi-infinite discrete random medium. For sparse media, we will use a somewhat different technique.

To reveal the difference between these two approaches, we consider the following Foldy–Lax model for the scattering by N particles randomly distributed throughout a volume V [1]:

$$u = u_0 + \sum_i \mathcal{A}_{0i} u_i, \tag{1}$$

$$u_i = u_{0i} + \sum_{j\neq i} \mathcal{A}_{ij} u_j. \tag{2}$$

In Eqs. (1) and (2), u is the total field, u_0 is the incident field, u_i is the field exciting particle i centered at $\mathbf{R}_i$, $\mathcal{A}_{0i} = \mathcal{A}_{0i}(\mathbf{r}, \mathbf{R}_i)$ is a (linear) operator describing the scattering from particle i to the observation point $\mathbf{r}$, u_{0i} is the incident field at the origin of particle i, and the operator $\mathcal{A}_{ij} = \mathcal{A}_{ij}(\mathbf{R}_i, \mathbf{R}_j)$ describes the scattering from particle j to particle i. Note that the vectors $\mathbf{r}$ and $\mathbf{R}_i$ extend from the common origin O of the laboratory coordinate system. Taking the configuration average of Eqs. (1) and (2) under the assumptions that the positions of all the particles are equally probable within the volume V, and that the Lax quasi-crystalline approximation $\langle u_j \rangle_{ij} = \langle u_j \rangle_j$ applies, yields

$$\langle u \rangle = u_0 + n_0 \int \mathcal{A}_{0i} \langle u_i \rangle_i \, \mathrm{d}^3\mathbf{R}_i, \tag{3}$$

$$\langle u_i \rangle_i = u_{0i} + n_0 \int \mathcal{A}_{ij} \langle u_j \rangle_j g(R_{ij}) \, \mathrm{d}^3\mathbf{R}_j, \tag{4}$$

where $n_0 = N/V$ is the particle number concentration, $\langle u_i \rangle_i$ is the conditional configuration average of u_i with the position of particle i held fixed, $\langle u_i \rangle_{ij}$ is the conditional configuration average of u_i with the positions of particles i and j held fixed, and $g(R_{ij}) = g(\mathbf{R}_{ij}) = g(\mathbf{R}_i, \mathbf{R}_j)$ with $\mathbf{R}_{ij} = \mathbf{R}_i - \mathbf{R}_j$ is the pair correlation function. For dense media, we first solve the integral equation (4) for $\langle u_i \rangle_i$, and then compute $\langle u \rangle$ from the integral representation (3). For sparse media, we consider the iterated solution of Eq. (4) with $g(R_{ij}) = 1$,

$$\langle u_i \rangle_i = u_{0i} + n_0 \int \mathcal{A}_{ij} u_{0i} \, \mathrm{d}^3\mathbf{R}_j + n_0^2 \int \mathcal{A}_{ij} \mathcal{A}_{jk} u_{0k} \, \mathrm{d}^3\mathbf{R}_k \mathrm{d}^3\mathbf{R}_j + \cdots, \tag{5}$$

sum up the series (5) (assuming that it converges), and compute $\langle u \rangle$ from Eq. (3).

For sparse media, an alternative approach is the following [7–9]: (i) consider the iterated solution of Eq. (2) under the Twersky approximation,

$$u_i = u_{0i} + \sum_{j\neq i} \mathcal{A}_{ij} u_{0j} + \sum_{j\neq i} \sum_{k\neq i,j} \mathcal{A}_{ij} \mathcal{A}_{jk} u_{0k} + \cdots; \tag{6}$$

(ii) insert Eq. (6) in Eq. (1) and obtain an order-of-scattering expansion for the total field

$$u = u_0 + \sum_i \mathcal{A}_{0i} u_{0i} + \sum_i \sum_{j\neq i} \mathcal{A}_{0i} \mathcal{A}_{ij} u_{0j} + \sum_i \sum_{j\neq i} \sum_{k\neq i,j} \mathcal{A}_{0i} \mathcal{A}_{ij} \mathcal{A}_{jk} u_{0ki} \cdots; \tag{7}$$

(iii) take the configuration average of the expansion (7) under the assumption of independent particle positions and get

$$\langle u \rangle = u_0 + n_0 \int \mathcal{A}_{0i} u_{0i} \, \mathrm{d}^3\mathbf{R}_i + n_0^2 \int \mathcal{A}_{0i} \mathcal{A}_{ij} u_{0j} \, \mathrm{d}^3\mathbf{R}_j \mathrm{d}^3\mathbf{R}_i + n_0^3 \int \mathcal{A}_{0i} \mathcal{A}_{ij} \mathcal{A}_{jk} u_{0k} \, \mathrm{d}^3\mathbf{R}_k \mathrm{d}^3\mathbf{R}_j \mathrm{d}^3\mathbf{R}_i + \cdots; \tag{8}$$

and finally, (iv) sum up the series (8).

Our paper is organized as follows. In Section 2, we summarize the results of Ref. [1]. Section 3 deals with the computation of the coherent field for a dense distribution of particles. Specifically, the coherent fields reflected and transmitted by the layer, and the coherent field inside the layer will be computed. For the latter, several solution methods based on the approaches by Waterman and Truell [2], and Tsang and Kong [6], as well as on a simplified approach relying on a so called sparse-medium approximation for the integration domain, will be examined. The results are then particularized to the cases of normal incidence and a semi-infinite discrete random medium. In Section 4 we present a self-consistent derivation of the coherent field for a sparse distribution of particles, and discuss the Twersky and Foldy approximations. The paper is concluded with a short discussion.

2. Recapitulation of the results of Ref. [1].

In this section, we summarize the results established in Ref. [1].

We consider the problem of electromagnetic scattering by a discrete random medium. More specifically, we consider a group of N identical spherical particles of radius a centered at quasi-random positions $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N \in D$, where the domain D is a laterally infinite plane-parallel layer with imaginary (non-scattering) boundaries $z = 0$ and $z = H$. The wavenumbers of the non-absorbing, non-magnetic background medium and the particles are k_1 and $k_2 = \mathrm{m}k_1$, respectively, where m is the relative refractive index. We denote by $f = n_0 V_0$ the particle volume concentration, where $n_0 = N/V$ is their number concentration, V is the cumulative volume occupied by the particles, and $V_0 = (4/3)\pi a^3$ is the volume of each particle. The particulate medium is illuminated by a plane electromagnetic wave with the propagation direction given by the unit vector $\widehat{\mathbf{s}} = \widehat{\mathbf{s}}(\theta_0, \varphi_0)$ and the amplitude $\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}})$, that is,

$$\mathbf{E}_0(\mathbf{r}) = \boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}) \mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{s}}\cdot\mathbf{r}}, \tag{9}$$

$$\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}) = \mathcal{E}_{0\theta} \widehat{\boldsymbol{\theta}}(\widehat{\mathbf{s}}) + \mathcal{E}_{0\varphi} \widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{s}}), \tag{10}$$

where $\mathrm{j} = \sqrt{-1}$; θ and φ hereinafter denote the corresponding spherical-coordinate angles; and $\mathcal{E}_{0\theta}$ and $\mathcal{E}_{0\varphi}$ are the polarized components of the amplitude vector.

Under the quasi-crystalline approximation, the conditional configuration-averaged exciting field coefficients $\langle \mathrm{e}_i \rangle_i (\mathbf{R}_i)$ satisfy the integral equation

$$\langle \mathrm{e}_i \rangle_i (\mathbf{R}_i) = \mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}}\cdot\mathbf{R}_i} \mathrm{e}_0 + n_0 \int_{D - D_{2a}(\mathbf{R}_i)} \mathrm{Q}(k_1 \mathbf{R}_{ij}) \langle \mathrm{e}_j \rangle_j (\mathbf{R}_j) g(R_{ij}) \, \mathrm{d}^3\mathbf{R}_j, \tag{11}$$

where $D_{2a}(\mathbf{R}_i)$ is a complete or a truncated sphere of radius $2a$ centered at the origin of particle i; $\mathbf{k}_{1\mathrm{s}} = k_1 \widehat{\mathbf{s}} = \mathbf{k}_{1\mathrm{s}\perp} + k_{1\mathrm{z}}(\mathbf{k}_{1\mathrm{s}\perp}) \widehat{\mathbf{z}}$ is the wave vector of the incident field; $k_{1\mathrm{z}}(\mathbf{k}_{1\mathrm{s}\perp}) = \sqrt{k_1^2 - k_{1\mathrm{s}\perp}^2}$; $\mathrm{e}_0 = 4\pi \boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}) \cdot \mathbf{x}^\star(\widehat{\mathbf{s}})$ is the vector of the incident field coefficients; $\mathbf{x}(\widehat{\mathbf{r}}) = [(-\mathrm{j})^n \mathbf{m}_{mn}(\widehat{\mathbf{r}}), \mathrm{j}(-\mathrm{j})^n \mathbf{n}_{mn}(\widehat{\mathbf{r}})]^T$ is a vector concatenating the vector spherical harmonics $\mathbf{m}_{mn}(\widehat{\mathbf{r}})$ and

$\mathbf{n}_{mn}(\widehat{\mathbf{r}})$, $\mathsf{Q}(k_1\mathbf{R}_{ij}) = \mathcal{T}_{31}^T(k_1\mathbf{R}_{ij})\mathsf{T}$; $\mathcal{T}_{31}(k_1\mathbf{R}_{ij})$ is the translation matrix relating the vectors of radiating and regular vector spherical wave functions $\mathbf{X}_3(k_1\mathbf{r}_j)$ and $\mathbf{X}_1(k_1\mathbf{r}_i)$, respectively, that is, $\mathbf{X}_3(k_1\mathbf{r}_j) = \mathcal{T}_{31}(k_1\mathbf{R}_{ij})\mathbf{X}_1(k_1\mathbf{r}_i)$ for $\mathbf{r}_j = \mathbf{r}_i + \mathbf{R}_{ij}$ and $r_i < R_{ij}$; and T is the particle-centered transition matrix of a spherical particle. Also, the asterisk denotes a complex-conjugate value, while T stands for "transposed". For the definitions of the vector spherical harmonics and the vector spherical wave functions we refer to Appendix A of Ref. [1]. To solve the integral equation (11) we look for a solution in the form

$$\left\langle \mathsf{e}_i \right\rangle_i (\mathbf{R}_i) = \sum_{b=\pm} \mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{R}_i}\mathsf{e}_b, \tag{12}$$

where b stands for the signs + and −, and the vectors e_b are unknown and have to be determined. In terms of the effective wavenumber in the particulate medium K, the wave vectors $\mathbf{K}_b$ in Eq. (12) are defined by $\mathbf{K}_b = K\widehat{\mathbf{s}}_b = \mathbf{K}_\perp + bK_z(\mathbf{K}_\perp)\widehat{\mathbf{z}}$, where $K_z(\mathbf{K}_\perp) = \sqrt{K^2 - K_\perp^2}$, $\widehat{\mathbf{s}}_+ = \widehat{\mathbf{s}}_\mathrm{T} = \widehat{\mathbf{s}}_\mathrm{T}(\theta_\mathrm{T}, \varphi_\mathrm{T})$ is the direction of the transmitted wave, $\widehat{\mathbf{s}}_- = \widehat{\mathbf{s}}_\mathrm{TR} = \widehat{\mathbf{s}}_\mathrm{TR}(\theta_\mathrm{TR}, \varphi_\mathrm{T})$ is the direction of the transmitted wave reflected by the upper boundary, and $\theta_\mathrm{TR} = \pi - \theta_\mathrm{T}$. The analysis is performed under the assumptions that the representation (12) is valid for all z_i in D even in the critical domains $0 \le z_i < 2a$ and $H - 2a < z_i \le H$. The solution obtained under these assumptions is referred to as the zeroth-order solution. Substituting Eq. (12) in Eq. (11), integrating over the positions of the particles, separating the upward and downward propagating waves, and then balancing the waves with the wavenumbers k_1 and K, we are led to

1. two homogeneous systems of equations corresponding to the generalized Lorenz–Lorentz law:

$$\mathsf{e}_b = n_0(\mathsf{J}_{1a}^b + \mathsf{J}_{2a}^b)\mathsf{T}\mathsf{e}_b, \quad b = \pm; \tag{13}$$

2. two inhomogeneous systems of equations corresponding to the generalized Ewald–Oseen extinction theorem:

$$\mathsf{e}_0 + \sum_{b=\pm} n_0 \mathsf{J}_{2z0}^b \mathsf{T}\mathsf{e}_b = 0, \tag{14}$$

$$\sum_{b=\pm} \mathrm{e}^{\mathrm{j}bK_z(\mathbf{K}_\perp)H} \mathsf{J}_{2zH}^b \mathsf{T}\mathsf{e}_b = 0; \tag{15}$$

and

3. the Snell law: $K\sin\theta_\mathrm{T} = k_1 \sin\theta_0$ and $\varphi_\mathrm{T} = \varphi_0$.

The expressions for the matrices J_{1a}^b, J_{1a}^b, J_{2z0}^b, and J_{2zH}^b are given in Ref. [1].

For the η-polarized incident field

$$\mathbf{E}_{0\eta}(\mathbf{r}) = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}})\mathrm{e}^{\mathrm{j}k_1\widehat{\mathbf{s}}\cdot\mathbf{r}}, \tag{16}$$

the polarized components $\mathsf{e}_{i\eta}$ of e_i, defined through the relation $\mathsf{e}_i = \sum_{\eta=\theta,\varphi} \mathcal{E}_{0\eta}\mathsf{e}_{i\eta}$, satisfy the integral equation

$$\left\langle \mathsf{e}_{i\eta} \right\rangle_i = \mathrm{e}^{\mathrm{j}\mathbf{k}_{1s}\cdot\mathbf{R}_i}\mathsf{e}_{0\eta} + n_0 \int_{D-D_{2a}(\mathbf{R}_i)} \mathsf{Q}(k_1\mathbf{R}_{ij}) \left\langle \mathsf{e}_{j\eta} \right\rangle_j g(R_{ij})\mathrm{d}^3\mathbf{R}_j. \tag{17}$$

In Eq. (17), we have $\mathsf{e}_{0\eta} = 4\pi \mathsf{x}_\eta^\star(\widehat{\mathbf{s}})$ and $\mathbf{x}(\widehat{\mathbf{r}}) = \sum_{\eta=\theta,\varphi} \mathsf{x}_\eta(\widehat{\mathbf{r}})\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{r}})$, meaning that the components $e_{0\eta mn}^1$ and $e_{0\eta mn}^2$ of the vector $\mathsf{e}_{0\eta} = [e_{0\eta mn}^1, e_{0\eta mn}^2]^T$ are computed as

$$e_{0\eta mn}^1 = 4\pi \mathrm{j}^n \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}) \cdot \mathbf{m}_{-mn}(\widehat{\mathbf{s}}), \tag{18}$$

$$e_{0\eta mn}^2 = -4\pi \mathrm{j}^{n+1} \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}) \cdot \mathbf{n}_{-mn}(\widehat{\mathbf{s}}). \tag{19}$$

As in Eq. (12), the solution of the above integral equation is sought in the form

$$\left\langle \mathsf{e}_{i\eta} \right\rangle_i = \sum_{b=\pm} \mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{R}_i}\mathsf{e}_{b\eta}, \tag{20}$$

in which case, the equations of the generalized Lorenz–Lorentz law and the generalized Ewald–Oseen extinction theorem read as

$$\mathsf{e}_{b\eta} = n_0(\mathsf{J}_{1a}^b + \mathsf{J}_{2a}^b)\mathsf{T}\mathsf{e}_{b\eta}, \quad b = \pm \tag{21}$$

and

$$\mathsf{e}_{0\eta} + \sum_{b=\pm} n_0 \mathsf{J}_{2z0}^b \mathsf{T}\mathsf{e}_{b\eta} = 0, \tag{22}$$

$$\sum_{b=\pm} \mathrm{e}^{\mathrm{j}bK_zH} \mathsf{J}_{2zH}^b \mathsf{T}\mathsf{e}_{b\eta} = 0, \tag{23}$$

respectively. The subsequent procedure is to express the components $e_{b\eta mn}^1$ and $e_{b\eta mn}^2$ of the vector $\mathsf{e}_{b\eta} = [e_{b\eta mn}^1, e_{b\eta mn}^2]^T$ as

$$e_{b\eta mn}^1 = 4\pi \mathrm{j}^n \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b) \cdot \mathbf{m}_{-mn}(\widehat{\mathbf{s}}_b) x_{b\eta n}^1, \tag{24}$$

$$e_{b\eta mn}^2 = -4\pi \mathrm{j}^{n+1} \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b) \cdot \mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b) x_{b\eta n}^2, \tag{25}$$

and to solve Eqs. (21)–(23) for the azimuth-independent coefficients $x_{b\eta n}^1$ and $x_{b\eta n}^2$.

The homogeneous systems of equations of the generalized Lorenz–Lorentz law are used to derive a dispersion equation for the effective wavenumber K. In Ref. [1] it was shown that the two homogeneous systems of equations (21) are identical to the homogeneous system of equations for a semi-infinite discrete random medium at normal incidence, and that the dispersion equation is direction and polarization independent. After computing the effective wavenumber K, the components of the vectors $\mathsf{e}_{b\eta}$ can be expressed in terms of two arbitrary constants, one for $\mathsf{e}_{+\eta}$ and the other one for $\mathsf{e}_{-\eta}$. These two constants are determined from the two inhomogeneous systems of equations of the generalized Ewald–Oseen extinction theorem (22) and (23) provided that each system of equations reduces to a single scalar equation. To derive these scalar equations we used the addition theorem for vector spherical harmonics, which yields the relations (cf. Appendix B of Ref. [1])

$$\sum_m [\mathbf{m}_{mn}(\widehat{\mathbf{k}}) \otimes \mathbf{m}_{-mn}(\widehat{\mathbf{k}}')] \cdot \widehat{\boldsymbol{\theta}}(\widehat{\mathbf{k}}') = \chi_n \sqrt{n(n+1)} M_n(\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}') \widehat{\boldsymbol{\theta}}(\widehat{\mathbf{k}}), \tag{26}$$

$$\sum_m [\mathbf{n}_{mn}(\widehat{\mathbf{k}}) \otimes \mathbf{n}_{-mn}(\widehat{\mathbf{k}}')] \cdot \widehat{\boldsymbol{\theta}}(\widehat{\mathbf{k}}') = -\chi_n \sqrt{n(n+1)} N_n(\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}') \widehat{\boldsymbol{\theta}}(\widehat{\mathbf{k}}) \tag{27}$$

for a θ-polarized incidence, and

$$\sum_m [\mathbf{m}_{mn}(\widehat{\mathbf{k}}) \otimes \mathbf{m}_{-mn}(\widehat{\mathbf{k}}')] \cdot \widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{k}}') = -\chi_n \sqrt{n(n+1)} N_n(\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}') \widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{k}}), \tag{28}$$

$$\sum_m [\mathbf{n}_{mn}(\widehat{\mathbf{k}}) \otimes \mathbf{n}_{-mn}(\widehat{\mathbf{k}}')] \cdot \widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{k}}') = \chi_n \sqrt{n(n+1)} M_n(\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}') \widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{k}}) \tag{29}$$

for a φ-polarized incidence. In Eqs. (26)–(29), the coefficient χ_n is given by

$$\chi_n = \frac{1}{2\pi n(n+1)} \sqrt{\frac{2n+1}{2}}, \tag{30}$$

and for the directions $\widehat{\mathbf{k}} = \widehat{\mathbf{k}}(\theta, \varphi)$ and $\widehat{\mathbf{k}}' = \widehat{\mathbf{k}}'(\theta', \varphi)$, the functions $M_n(\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}')$ and $N_n(\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}')$ are given, respectively, by

$$M_n(\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}') = \pi_n^1(x), \tag{31}$$

$$N_n(\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}') = x\pi_n^1(x) - \sqrt{n(n+1)}P_n(x), \tag{32}$$

where $x = \widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}'$, $\pi_n^1(x) = P_n^1(x)/\sqrt{1-x^2}$, and $P_n^{|m|}$ are the normalized associated Legendre functions of degree n and order m. The scalar equations are found to be

$$1 = -\mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}} \sum_{b=\pm} \frac{b}{K_z - bk_{1z}} \sum_n \chi_n \sqrt{n(n+1)} \times \left[M_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_b) T_n^1 x_{b\theta n}^1 - N_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_b) T_n^2 x_{b\theta n}^2\right], \tag{33}$$

$$0 = \sum_{b=\pm} \frac{b}{K_z + bk_{1z}} \mathrm{e}^{\mathrm{j}bK_z H} \sum_n \chi_n \sqrt{n(n+1)} \times \left[M_n(\widehat{\mathbf{s}}_{\mathrm{R}} \cdot \widehat{\mathbf{s}}_b) T_n^1 x_{b\theta n}^1 - N_n(\widehat{\mathbf{s}}_{\mathrm{R}} \cdot \widehat{\mathbf{s}}_b) T_n^2 x_{b\theta n}^2\right] \tag{34}$$

for a θ-polarized incidence, and

$$1 = \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}} \sum_{b=\pm} \frac{b}{K_z - bk_{1z}} \sum_n \chi_n \sqrt{n(n+1)} \times \left[N_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_b) T_n^1 x_{b\varphi n}^1 - M_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_b) T_n^2 x_{b\varphi n}^2\right], \tag{35}$$

$$0 = \sum_{b=\pm} \frac{b}{K_z + bk_{1z}} \mathrm{e}^{\mathrm{j}bK_z H} \sum_{n'} \chi_n \sqrt{n(n+1)} \times \left[N_n(\widehat{\mathbf{s}}_{\mathrm{R}} \cdot \widehat{\mathbf{s}}_b) T_n^1 x_{b\varphi n}^1 - M_n(\widehat{\mathbf{s}}_{\mathrm{R}} \cdot \widehat{\mathbf{s}}_b) T_n^2 x_{b\varphi n}^2\right] \tag{36}$$

for a φ-polarized incidence. In Eqs. (33)–(36), K_z and k_{1z} are given by

$$K_z = K_z(\mathbf{k}_{1s\perp}) = \sqrt{K^2 - k_{1s\perp}^2} = \sqrt{K^2 - k_1^2 \sin^2\theta_0}, \tag{37}$$

and

$$k_{1z} = k_{1z}(\mathbf{k}_{1s\perp}) = \sqrt{k_1^2 - k_{1s\perp}^2} = \sqrt{k_1^2 - k_1^2 \sin^2\theta_0}, \tag{38}$$

respectively. Regarding the behavior of the coefficients $x_{b\eta n}^{1,2}$, the following two results have been established through a representative numerical analysis:

1. $|x_{-\eta n}^{1,2}| \ll |x_{+\eta n}^{1,2}|$, and
2. $x_{+n}^{1,2} \approx 1$ for small values of the volume concentration f, e.g., $f = 0.01$.

3. Coherent field in dense media

A rigorous method for computing the coherent field in a discrete random medium has been given by Waterman and Truell [2]. This method which takes into account whether the observation point is inside or outside the discrete random layer is summarized below.

If the observation point $\mathbf{r}$ is outside of any particle, the total field is the sum of the incident and all scattered fields, i.e., $\mathbf{E}_0(\mathbf{r}) + \sum_{i=1}^N \mathbf{E}_{\mathrm{sct}i}(\mathbf{r})$; while inside particle i, the total field is the internal field $\mathbf{E}_{\mathrm{int}i}(\mathbf{r})$ when excited by $\mathbf{E}_{\mathrm{exc}i}(\mathbf{r})$. Defining the indicator function

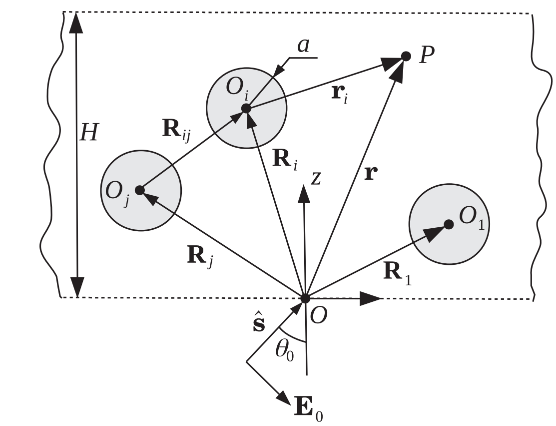

Fig. 1. Scattering by a layer of spherical particles.

$$\alpha(\mathbf{r} - \mathbf{R}_i) = \begin{cases} 0, & \mathbf{r} \in D_a(\mathbf{R}_i), \\ 1, & \mathbf{r} \notin D_a(\mathbf{R}_i), \end{cases} \tag{39}$$

where $D_a(\mathbf{R}_i)$ is a sphere of radius a centered at $\mathbf{R}_i$, we express this statement mathematically as

$$\mathbf{E}(\mathbf{r}) = \left[\prod_{i=1}^N \alpha(\mathbf{r} - \mathbf{R}_i)\right]\left[\mathbf{E}_0(\mathbf{r}) + \sum_{i=1}^N \mathbf{E}_{\mathrm{sct}i}(\mathbf{r})\right] + \sum_{i=1}^N [1 - \alpha(\mathbf{r} - \mathbf{R}_i)]\mathbf{E}_{\mathrm{int}i}(\mathbf{r}). \tag{40}$$

Using the identity

$$\prod_{i=1}^N \alpha(\mathbf{r} - \mathbf{R}_i) = 1 - \sum_{i=1}^N [1 - \alpha(\mathbf{r} - \mathbf{R}_i)], \tag{41}$$

and the relation giving the field exciting particle i, $\mathbf{E}_{\mathrm{exc}i}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \sum_{j\neq i}^N \mathbf{E}_{\mathrm{sct}j}(\mathbf{r})$ for $\mathbf{r} \in D_a(\mathbf{R}_i)$, we obtain

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \sum_{i=1}^N \mathbf{E}_{\mathrm{sct}i}(\mathbf{r}) + \sum_{i=1}^N [1 - \alpha(\mathbf{r} - \mathbf{R}_i)][\mathbf{E}_{\mathrm{int}i}(\mathbf{r}) - \mathbf{E}_{\mathrm{exc}i}(\mathbf{r})]. \tag{42}$$

Taking the configuration average of Eq. (42) over the particle positions (for the equations governing the configuration averaging process, we refer to Section 4 of Ref. [1]) we find that for an external observation point $\mathbf{r}$ located in the domains $z \le -a$ or $z \ge H + a$, the coherent field $\mathbf{E}_c(\mathbf{r}) = \langle \mathbf{E}(\mathbf{r}) \rangle$ is

$$\mathbf{E}_c(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + n_0 \int_D \langle \mathbf{E}_{\mathrm{sct}i}(\mathbf{r}) \rangle_i \mathrm{d}^3\mathbf{R}_i, \tag{43}$$

while for an external observation point residing in the critical domains $-a < z \le 0$ or $H \le z < H + a$, a truncated sphere of radius a should be excluded from the integration domain D (Fig. 2). For an internal observation point $\mathbf{r}$ situated in the domain $a \le z \le H - a$, the coherent field is

$$\mathbf{E}_c(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + n_0 \int_{D - D_a(\mathbf{r})} \langle \mathbf{E}_{\mathrm{sct}i}(\mathbf{r}) \rangle_i \mathrm{d}^3\mathbf{R}_i + n_0 \int_{D_a(\mathbf{r})} [\langle \mathbf{E}_{\mathrm{int}i}(\mathbf{r}) \rangle_i - \langle \mathbf{E}_{\mathrm{exc}i}(\mathbf{r}) \rangle_i] \mathrm{d}^3\mathbf{R}_i, \tag{44}$$

where $D_a(\mathbf{r})$ is a complete sphere of radius a centered at $\mathbf{r}$; for an internal observation point residing in the critical domains $0 \le z < a$ or $H - a < z \le H$, $D_a(\mathbf{r})$ is a truncated sphere of radius a (Fig. 3).

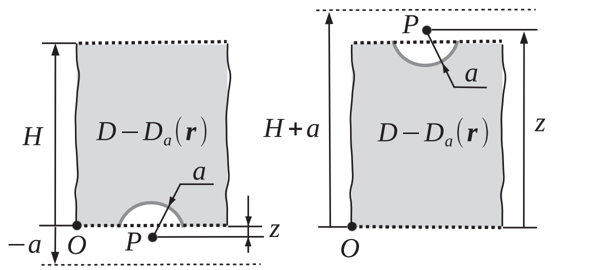

Fig. 2. For an external observation point P and for $-a<z\le 0$ or $H\le z<H+a$, the domain of integration is D with a truncated sphere of radius a excluded. O is the origin of the laboratory coordinate system.

In Eqs. (43) and (44), the conditional configuration averages of the fields are given by

$$\left\langle \mathbf{E}_{\mathrm{sct}i}(\mathbf{r})\right\rangle_i = \left\langle \mathbf{E}^{(i)}_{\mathrm{sct}i}(\mathbf{r}_i)\right\rangle_i = \mathbf{X}_3^T(k_1\mathbf{r}_i)\mathsf{T}\left\langle \mathsf{e}_i\right\rangle_i, \tag{45}$$

$$\left\langle \mathbf{E}_{\mathrm{exc}i}(\mathbf{r})\right\rangle_i = \left\langle \mathbf{E}^{(i)}_{\mathrm{exc}i}(\mathbf{r}_i)\right\rangle_i = \mathbf{X}_1^T(k_1\mathbf{r}_i)\left\langle \mathsf{e}_i\right\rangle_i, \tag{46}$$

$$\left\langle \mathbf{E}_{\mathrm{int}i}(\mathbf{r})\right\rangle_i = \left\langle \mathbf{E}^{(i)}_{\mathrm{int}i}(\mathbf{r}_i)\right\rangle_i = \mathbf{X}_1^T(k_2\mathbf{r}_i)\mathsf{T}_{\mathrm{int}}\left\langle \mathsf{e}_i\right\rangle_i, \tag{47}$$

where $\mathbf{r}_i=\mathbf{r}-\mathbf{R}_i$, $\left\langle \mathsf{e}_i\right\rangle_i$ is the solution of the integral equation (11), and, for a spherical particle, $\mathsf{T}_{\mathrm{int}}$ is the particle-centered "transition matrix" relating the expansion coefficients of the internal field to those of the exciting field.

The first integral in Eq. (44) corresponds to the configurations in which the observation point $\mathbf{r}$ is external to all particles. The first term in the second integral of Eq. (44) gives the internal contribution (the point $\mathbf{r}$ is inside the particle), while the second term cancels out the external point contribution since the probability that $\mathbf{r}$ is an outside point diminishes [2].

As for the exciting field coefficients, we will consider only the zeroth-order solution for the coherent field, in which case the exclusion domain $D_a(\mathbf{r})$ is a complete sphere of radius a. In particular, we will use Eq. (43) to compute the coherent fields reflected and transmitted by the layer, and Eq. (44) to compute the coherent field inside the layer. To integrate over all positions of particle i we use a local coordinate system centered at the observation point. In this regard, we make the change of variable $\mathbf{p}=\mathbf{R}_i-\mathbf{r}=-\mathbf{r}_i$, implying $\mathbf{p}=\mathbf{p}_\perp+p_z\widehat{\mathbf{z}}$ with $\mathbf{p}_\perp=\mathbf{R}_{i\perp}-\mathbf{r}_\perp$ and $p_z=z_i-z$, and use the symmetry relations (cf. Eq. (256) of Ref. [1])

$$\mathbf{M}^3_{mn}(-k_1\mathbf{p}) = (-1)^n\mathbf{M}^3_{mn}(k_1\mathbf{p}), \tag{48}$$

$$\mathbf{N}^3_{mn}(-k_1\mathbf{p}) = (-1)^{n+1}\mathbf{N}^3_{mn}(k_1\mathbf{p}). \tag{49}$$

It should be pointed out that the coherent field component given by the first integral in Eq. (44) alters the statistics of the problem slightly in the sense that the single particle probability density function $p(\mathbf{R}_i)=1/V$ is replaced by

$$p(\mathbf{R}_i;\mathbf{r}) = \frac{1}{V-V_0}\Theta_a(\mathbf{R}_i;\mathbf{r}),$$
$$\Theta_a(\mathbf{R}_i;\mathbf{r}) = \begin{cases} 1, & \mathbf{R}_i\in D-D_a(\mathbf{r}),\\ 0, & \text{otherwise.}\end{cases} \tag{50}$$

The new probability density function (50), which is similar to the hole-correction approximation, can be regarded as a conditional probability given the position of the observation point $\mathbf{r}$; it is also normalized, and we have

$$\sum_i\int_D p(\mathbf{R}_i;\mathbf{r})\left\langle \mathbf{E}_{\mathrm{sct}i}(\mathbf{r})\right\rangle_i \mathrm{d}^3\mathbf{R}_i = \frac{N}{V-V_0}\int_{D-D_a(\mathbf{r})}\left\langle \mathbf{E}_{\mathrm{sct}i}(\mathbf{r})\right\rangle_i \mathrm{d}^3\mathbf{R}_i$$
$$\approx n_0\int_{D-D_a(\mathbf{r})}\left\langle \mathbf{E}_{\mathrm{sct}i}(\mathbf{r})\right\rangle_i \mathrm{d}^3\mathbf{R}_i.$$

3.1. The coherent field reflected by the layer

The coherent field reflected by the layer is

$$\mathbf{E}_{\mathrm{R}\eta}(\mathbf{r}) = n_0\sum_{b=\pm}\sum_{mn}\int_D\Big[T_n^1 e^1_{b\eta mn}\mathbf{M}^3_{mn}(k_1\mathbf{r}_i) + T_n^2 e^2_{b\eta mn}\mathbf{N}^3_{mn}(k_1\mathbf{r}_i)\Big]\mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{R}_i}\mathrm{d}^3\mathbf{R}_i,\quad z\le -a. \tag{51}$$

For $z\le -a<0\le z_i$, we use the integral representations of the radiating vector spherical wave functions in terms of plane electromagnetic waves, that is ($\mathbf{p}=\mathbf{R}_i-\mathbf{r}=-\mathbf{r}_i$) [10]

$$\mathbf{M}^3_{mn}(k_1\mathbf{p}) = \frac{1}{2\pi\mathrm{j}^n}\int \mathbf{m}_{mn}(\widehat{\mathbf{k}}^+)\mathrm{e}^{\mathrm{j}[\mathbf{k}_\perp\cdot\mathbf{p}_\perp+k_{1z}(\mathbf{k}_\perp)(z_i-z)]}\frac{\mathrm{d}^2\mathbf{k}_\perp}{k_1k_{1z}(\mathbf{k}_\perp)}, \tag{52}$$

$$\mathbf{N}^3_{mn}(k_1\mathbf{p}) = \frac{1}{2\pi\mathrm{j}^n}\int \mathrm{j}\mathbf{n}_{mn}(\widehat{\mathbf{k}}^+)\mathrm{e}^{\mathrm{j}[\mathbf{k}_\perp\cdot\mathbf{p}_\perp+k_{1z}(\mathbf{k}_\perp)(z_i-z)]}\frac{\mathrm{d}^2\mathbf{k}_\perp}{k_1k_{1z}(\mathbf{k}_\perp)}, \tag{53}$$

$$\mathbf{k}^+ = \mathbf{k}_\perp + k_{1z}(\mathbf{k}_\perp)\widehat{\mathbf{z}}, \tag{54}$$

to obtain

$$\int_D \mathbf{M}^3_{mn}(k_1\mathbf{r}_i)\mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{R}_i}\mathrm{d}^3\mathbf{R}_i = -\frac{2\pi}{\mathrm{j}^{n+1}}\frac{1-\mathrm{e}^{\mathrm{j}(bK_z+k_{1z})H}}{k_1k_{1z}(bK_z+k_{1z})}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{R}}\cdot\mathbf{r}}\mathbf{m}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}}), \tag{55}$$

$$\int_D \mathbf{N}^3_{mn}(k_1\mathbf{r}_i)\mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{R}_i}\mathrm{d}^3\mathbf{R}_i = -\frac{2\pi}{\mathrm{j}^{n+1}}\frac{1-\mathrm{e}^{\mathrm{j}(bK_z+k_{1z})H}}{k_1k_{1z}(bK_z+k_{1z})}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{R}}\cdot\mathbf{r}}\mathrm{j}\mathbf{n}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}}), \tag{56}$$

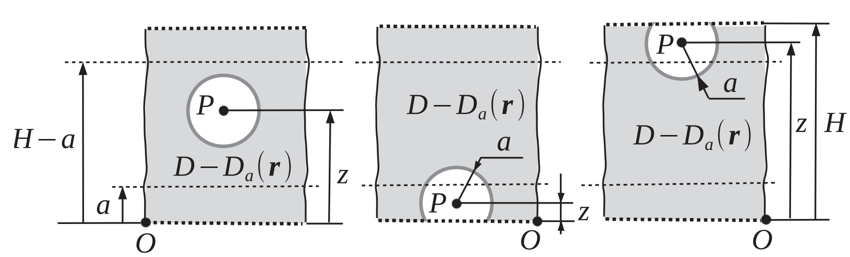

Fig. 3. For an internal observation point P, the exclusion domain $D_a(\mathbf{r})$ is a complete sphere of radius a if $a\le z\le H-a$ (left), and a truncated sphere of radius a if $0\le z<a$ (middle) or $H-a<z\le H$ (right).

with $\mathbf{k}_{1\mathrm{R}} = k_1 \widehat{\mathbf{s}}_\mathrm{R} = \mathbf{k}_{1\mathrm{s}\perp} - k_{1\mathrm{z}}\widehat{\mathbf{z}}$. Inserting Eqs. (24), (25), (55) and (56) in Eq. (51), we find that the coherent reflected field is computed according to

$$\begin{aligned}\mathbf{E}_{\mathrm{R}\eta}(\mathbf{r}) &= \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1\mathrm{z}}} \mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{R}}\cdot\mathbf{r}} \sum_{b=\pm} \frac{b}{K_\mathrm{z} + bk_{1\mathrm{z}}}[1 - \mathrm{e}^{\mathrm{j}(bK_\mathrm{z}+k_{1\mathrm{z}})H}] \\ &\times \sum_{mn}\{[\mathbf{m}_{mn}(\widehat{\mathbf{s}}_\mathrm{R}) \otimes \mathbf{m}_{-mn}(\widehat{\mathbf{s}}_b)] \cdot \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b) T_n^1 x_{b\eta n}^1 \\ &+ [\mathbf{n}_{mn}(\widehat{\mathbf{s}}_\mathrm{R}) \otimes \mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)] \cdot \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b) T_n^2 x_{b\eta n}^2\}. \end{aligned} \tag{57}$$

Defining the reflection coefficient R_η^0 through the relation

$$\mathbf{E}_{\mathrm{R}\eta}(\mathbf{r}) = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_\mathrm{R}) R_\eta^0 \mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{R}}\cdot\mathbf{r}} = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_\mathrm{R}) R_\eta^0 \mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}\perp}\cdot\mathbf{r}} \mathrm{e}^{-\mathrm{j}k_{1\mathrm{z}}z}, \tag{58}$$

and using the addition theorem for vector spherical harmonics (cf. Eqs. (26) and (27) and Eqs. (28) and (29)), we obtain

$$\begin{aligned} R_\theta^0 &= \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1\mathrm{z}}} \sum_{b=\pm} \frac{b}{K_\mathrm{z} + bk_{1\mathrm{z}}}\left[1 - \mathrm{e}^{\mathrm{j}(bK_\mathrm{z}+k_{1\mathrm{z}})H}\right] \sum_n \chi_n \sqrt{n(n+1)} \\ &\times [M_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_b) T_n^1 x_{b\theta n}^1 - N_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_b) T_n^2 x_{b\theta n}^2] \end{aligned} \tag{59}$$

for a θ-polarized incidence, and

$$\begin{aligned} R_\varphi^0 &= -\mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1\mathrm{z}}} \sum_{b=\pm} \frac{b}{K_\mathrm{z} + bk_{1\mathrm{z}}}\left[1 - \mathrm{e}^{\mathrm{j}(bK_\mathrm{z}+k_{1\mathrm{z}})H}\right] \sum_n \chi_n \sqrt{n(n+1)} \\ &\times [N_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_b) T_n^1 x_{b\varphi n}^1 - M_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_b) T_n^2 x_{b\varphi n}^2] \end{aligned} \tag{60}$$

for a φ-polarized incidence. Furthermore, from Eqs. (34)–(36) of the generalized Ewald–Oseen extinction theorem, we find in a first step

$$\begin{aligned} R_\theta^0 &= \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1\mathrm{z}}} \sum_{b=\pm} \frac{b}{K_\mathrm{z} + bk_{1\mathrm{z}}} \sum_n \chi_n \sqrt{n(n+1)} \\ &\times [M_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_b) T_n^1 x_{b\theta n}^1 - N_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_b) T_n^2 x_{b\theta n}^2], \end{aligned} \tag{61}$$

$$\begin{aligned} R_\varphi^0 &= -\mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1\mathrm{z}}} \sum_{b=\pm} \frac{b}{K_\mathrm{z} + bk_{1\mathrm{z}}} \sum_n \chi_n \sqrt{n(n+1)} \\ &\times [N_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_b) T_n^1 x_{b\varphi n}^1 - M_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_b) T_n^2 x_{b\varphi n}^2], \end{aligned} \tag{62}$$

and in a second step,

$$\begin{aligned} R_\theta^0 &= \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1\mathrm{z}}(K_\mathrm{z} + k_{1\mathrm{z}})}\left(1 - \mathrm{e}^{2\mathrm{j}K_\mathrm{z}H}\right) \sum_n \chi_n \sqrt{n(n+1)} \\ &\times [M_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_\mathrm{T}) T_n^1 x_{+\theta n}^1 - N_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_\mathrm{T}) T_n^2 x_{+\theta n}^2], \end{aligned} \tag{63}$$

$$\begin{aligned} R_\varphi^0 &= -\mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1\mathrm{z}}(K_\mathrm{z} + k_{1\mathrm{z}})}\left(1 - \mathrm{e}^{2\mathrm{j}K_\mathrm{z}H}\right) \sum_n \chi_n \sqrt{n(n+1)} \\ &\times [N_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_\mathrm{T}) T_n^1 x_{+\varphi n}^1 - M_n(\widehat{\mathbf{s}}_\mathrm{R}\cdot\widehat{\mathbf{s}}_\mathrm{T}) T_n^2 x_{+\varphi n}^2]. \end{aligned} \tag{64}$$

For a θ-polarized incidence, we plot in Fig. 4 the reflection coefficient $|R_\theta^0|$ together with the reflection coefficient $|R_{\theta\mathrm{EFA}}^0|$ for a homogeneous layer with thickness H and wavenumber K placed in a background medium with wavenumber k_1. The latter corresponds to the effective field approximation, and is computed as

$$R_{\eta\mathrm{EFA}}^0 = \frac{(1 - \mathrm{e}^{2\mathrm{j}K_\mathrm{z}H}) r_\eta}{1 - (r_\eta)^2 \mathrm{e}^{2\mathrm{j}K_\mathrm{z}H}}, \tag{65}$$

where

$$r_\eta = \begin{cases} \dfrac{K\cos\theta_0 - k_1\cos\theta}{K\cos\theta_0 + k_1\cos\theta}, & \eta = \theta, \\[2ex] \dfrac{k_1\cos\theta_0 - K\cos\theta}{k_1\cos\theta_0 + K\cos\theta}, & \eta = \varphi. \end{cases} \tag{66}$$

The results show that (i) in general and in agreement with the findings of Ref. [11], the reflection coefficients are small; (ii) they increase with the volume concentration; and (iii) in the low frequency limit, the effective field approximation is valid.

3.2. The coherent field transmitted by the layer

The coherent field transmitted by the layer is

$$\begin{aligned} \mathbf{E}_{\mathrm{T}\eta}(\mathbf{r}) &= \mathbf{E}_{0\eta}(\mathbf{r}) + n_0 \sum_{b=\pm}\sum_{mn} \int_D \Big[T_n^1 e_{b\eta mn}^1 \mathbf{M}_{mn}^3(k_1\mathbf{r}_i) \\ &+ T_n^2 e_{b\eta mn}^2 \mathbf{N}_{mn}^3(k_1\mathbf{r}_i)\Big] \mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{R}_i} \mathrm{d}^3\mathbf{R}_i, \quad z \ge H + a. \end{aligned} \tag{67}$$

For $z \ge H + a > H \ge z_i$, we use the integral representations ($\mathbf{p} = \mathbf{R}_i - \mathbf{r} = -\mathbf{r}_i$)

$$\mathbf{M}_{mn}^3(k_1\mathbf{p}) = \frac{1}{2\pi \mathrm{j}^n} \int \mathbf{m}_{mn}(\widehat{\mathbf{k}}^-) \mathrm{e}^{\mathrm{j}[\mathbf{k}_\perp\cdot\mathbf{p}_\perp - k_{1\mathrm{z}}(\mathbf{k}_\perp)(z_i - z)]} \frac{\mathrm{d}^2\mathbf{k}_\perp}{k_1 k_{1\mathrm{z}}(\mathbf{k}_\perp)}, \tag{68}$$

$$\mathbf{N}_{mn}^3(k_1\mathbf{p}) = \frac{1}{2\pi \mathrm{j}^n} \int \mathrm{j}\mathbf{n}_{mn}(\widehat{\mathbf{k}}^-) \mathrm{e}^{\mathrm{j}[\mathbf{k}_\perp\cdot\mathbf{p}_\perp - k_{1\mathrm{z}}(\mathbf{k}_\perp)(z_i - z)]} \frac{\mathrm{d}^2\mathbf{k}_\perp}{k_1 k_{1\mathrm{z}}(\mathbf{k}_\perp)}, \tag{69}$$

$$\mathbf{k}^- = \mathbf{k}_\perp - k_{1\mathrm{z}}(\mathbf{k}_\perp)\widehat{\mathbf{z}}, \tag{70}$$

yielding

$$\begin{aligned} &\int_D \mathbf{M}_{mn}^3(k_1\mathbf{r}_i) \mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{R}_i} \mathrm{d}^3\mathbf{R}_i \\ &= -\frac{2\pi}{\mathrm{j}^{n+1}} \frac{1 - \mathrm{e}^{\mathrm{j}(bK_\mathrm{z} - k_{1\mathrm{z}})H}}{k_1 k_{1\mathrm{z}}(bK_\mathrm{z} - k_{1\mathrm{z}})} \mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}}\cdot\mathbf{r}} \mathbf{m}_{mn}(\widehat{\mathbf{s}}), \end{aligned} \tag{71}$$

$$\begin{aligned} &\int_D \mathbf{N}_{mn}^3(k_1\mathbf{r}_i) \mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{R}_i} \mathrm{d}^3\mathbf{R}_i \\ &= -\frac{2\pi}{\mathrm{j}^{n+1}} \frac{1 - \mathrm{e}^{\mathrm{j}(bK_\mathrm{z} - k_{1\mathrm{z}})H}}{k_1 k_{1\mathrm{z}}(bK_\mathrm{z} - k_{1\mathrm{z}})} \mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}}\cdot\mathbf{r}} \mathrm{j}\mathbf{n}_{mn}(\widehat{\mathbf{s}}), \end{aligned} \tag{72}$$

along with Eqs. (24) and (25) to obtain

$$\begin{aligned} \mathbf{E}_{\mathrm{T}\eta}(\mathbf{r}) &= \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}) \mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}}\cdot\mathbf{r}} + \mathrm{j}\frac{8\pi^2 n_0}{kk_{1\mathrm{z}}} \mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}}\cdot\mathbf{r}} \\ &\times \sum_{b=\pm} \frac{b}{K_\mathrm{z} - bk_{1\mathrm{z}}}\left[1 - \mathrm{e}^{\mathrm{j}(bK_\mathrm{z} - k_{1\mathrm{z}})H}\right] \\ &\times \sum_{mn}\{[\mathbf{m}_{mn}(\widehat{\mathbf{s}}) \otimes \mathbf{m}_{-mn}(\widehat{\mathbf{s}}_b)] \cdot \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b) T_n^1 x_{b\eta n}^1 \\ &+ [\mathbf{n}_{mn}(\widehat{\mathbf{s}}) \otimes \mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)] \cdot \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b) T_n^2 x_{b\eta n}^2\}. \end{aligned} \tag{73}$$

Defining the transmission coefficient T_η^2 by

$$\mathbf{E}_{\mathrm{T}\eta}(\mathbf{r}) = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}) T_\eta^2 \mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}}\cdot\mathbf{r}} = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}) T_\eta^2 \mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}\perp}\cdot\mathbf{r}} \mathrm{e}^{\mathrm{j}k_{1\mathrm{z}}z}, \tag{74}$$

and employing the addition theorem for vector spherical harmonics (cf. Eqs. (26) and (27) and Eqs. (28) and (29)), we get

$$\begin{aligned} T_\theta^2 &= 1 + \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1\mathrm{z}}} \sum_{b=\pm} \frac{b}{K_\mathrm{z} - bk_{1\mathrm{z}}}\left[1 - \mathrm{e}^{\mathrm{j}(bK_\mathrm{z} - k_{1\mathrm{z}})H}\right] \\ &\times \sum_n \chi_n \sqrt{n(n+1)}\Big[M_n(\widehat{\mathbf{s}}\cdot\widehat{\mathbf{s}}_b) T_n^1 x_{b\theta n}^1 - N_n(\widehat{\mathbf{s}}\cdot\widehat{\mathbf{s}}_b) T_n^2 x_{b\theta n}^2\Big] \end{aligned} \tag{75}$$

for a θ-polarized incidence, and

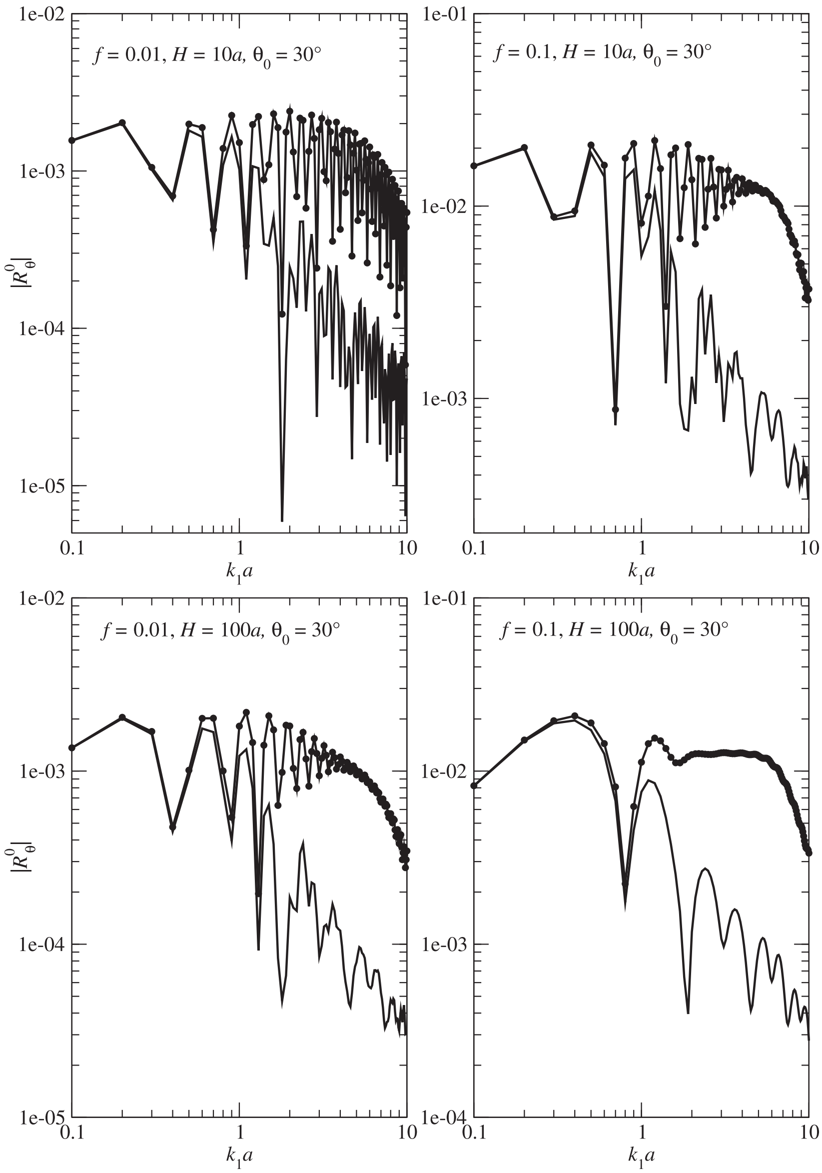

Fig. 4. Reflection coefficients $|R_\theta^0|$ (solid curve) and $|R_{\theta\mathrm{EFA}}^0|$ (solid curve with circles) for a θ-polarized incidence as functions of the size parameter $k_1 a$ and for two values of the volume concentration f and the layer thickness H. The incidence angle is $\theta_0 = 30^\circ$, the wavenumber of the background medium is $k_1 = 10\,\mu\mathrm{m}^{-1}$, the relative refractive index of the particles is $\mathrm{m} = 1.33$, and the maximum expansion order in the expansion (61) is $N_{\mathrm{rank}} = 15$.

$$T_\varphi^2 = 1 - \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}} \sum_{b=\pm} \frac{b}{K_z - bk_{1z}} \left[1 - \mathrm{e}^{\mathrm{j}(bK_z - k_{1z})H}\right] \times \sum_n \chi_n \sqrt{n(n+1)} \left[N_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_b) T_n^1 x_{b\varphi n}^1 - M_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_b) T_n^2 x_{b\varphi n}^2\right] \tag{76}$$

for a φ-polarized incidence. Furthermore, invoking Eqs. (33)–(35) of the generalized Ewald–Oseen extinction theorem, we find

$$T_\theta^2 = -\mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}} \sum_{b=\pm} \frac{b}{K_z - bk_{1z}} \mathrm{e}^{\mathrm{j}(bK_z - k_{1z})H} \times \sum_n \chi_n \sqrt{n(n+1)} \left[M_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_b) T_n^1 x_{b\theta n}^1 - N_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_b) T_n^2 x_{b\theta n}^2\right], \tag{77}$$

$$T_\varphi^2 = \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}} \sum_{b=\pm} \frac{b}{K_z - bk_{1z}} \mathrm{e}^{\mathrm{j}(bK_z - k_{1z})H} \times \sum_n \chi_n \sqrt{n(n+1)} \left[N_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_b) T_n^1 x_{b\varphi n}^1 - M_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_b) T_n^2 x_{b\varphi n}^2\right], \tag{78}$$

and then

$$T_\theta^2 = \mathrm{e}^{-\mathrm{j}(K_z + k_{1z})H} \left\{1 + \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}(K_z - k_{1z})} \left(1 - \mathrm{e}^{2\mathrm{j}K_z H}\right) \times \sum_n \chi_n \sqrt{n(n+1)} \left[M_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_{\mathrm{T}}) T_n^1 x_{+\theta n}^1 - N_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_{\mathrm{T}}) T_n^2 x_{+\theta n}^2\right]\right\}, \tag{79}$$

$$T_\varphi^2 = \mathrm{e}^{-\mathrm{j}(K_z + k_{1z})H} \left\{1 + \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}(K_z - k_{1z})} \left(1 - \mathrm{e}^{2\mathrm{j}K_z H}\right) \times \sum_n \chi_n \sqrt{n(n+1)} \left[N_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_{\mathrm{T}}) T_n^1 x_{+\varphi n}^1 - M_n(\widehat{\mathbf{s}} \cdot \widehat{\mathbf{s}}_{\mathrm{T}}) T_n^2 x_{+\varphi n}^2\right]\right\}. \tag{80}$$

Note that although the representations (77) and (78) are mathematically equivalent with the representations (79) and (80),

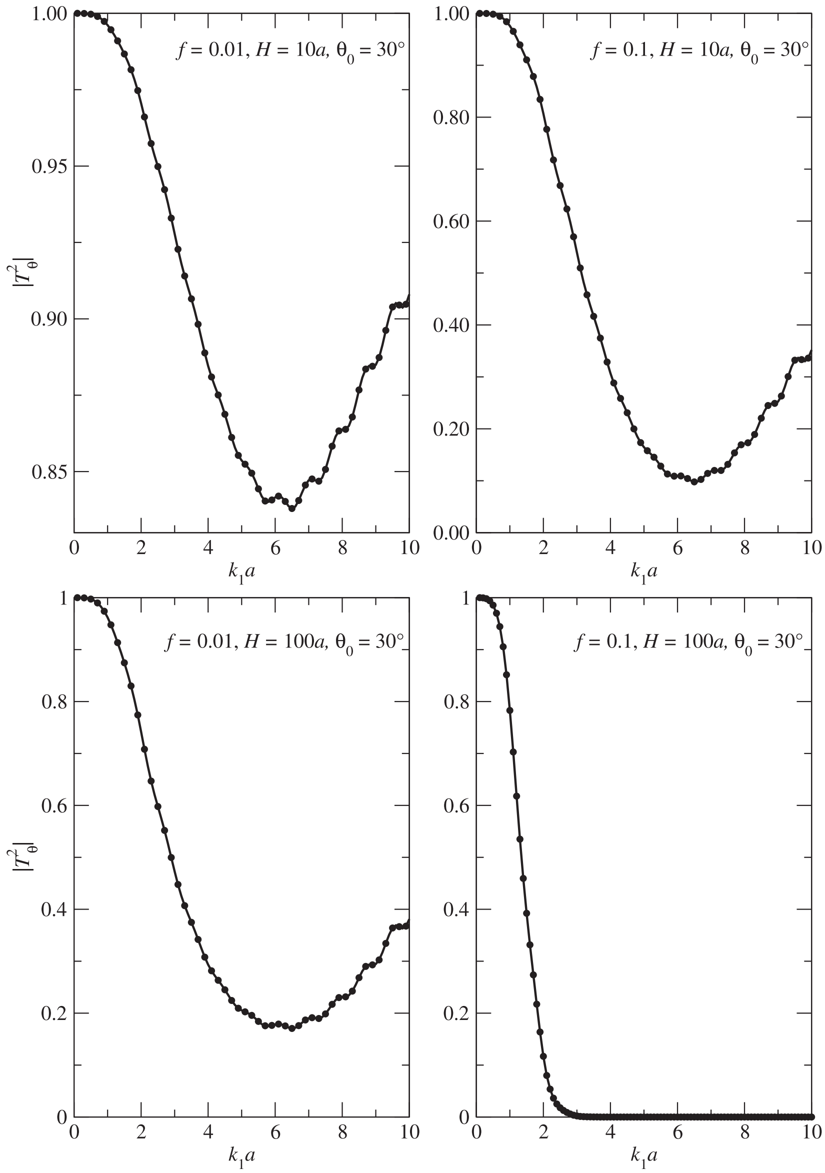

Fig. 5. Transmission coefficients $|T_\theta^2|$ (solid curve) and $|T_{\theta\mathrm{EFA}}^2|$ (solid curve with circles) for a θ-polarized incidence as functions of the size parameter k_1a and for two values of the volume concentration f and the layer thickness H. The incidence angle is $\theta_0 = 30^\circ$, the wavenumber of the background medium is $k_1 = 10\,\mu\mathrm{m}^{-1}$, the relative refractive index of the particles is $\mathrm{m} = 1.33$, and the maximum expansion order in the expansion (77) is $N_{\mathrm{rank}} = 15$.

respectively, the exponential term $\exp(-\mathrm{j}K_zH)$ shows that the last two are numerically unstable for large H. Therefore, in numerical implementations, the representations (77) and (78) should be used.

The plots in Fig. 5 show that the transmission coefficient $|T_\theta^2|$ agrees with the transmission coefficient $|T_{\theta\mathrm{EFA}}^2|$ corresponding to a homogeneous layer, where

$$T_{\eta\mathrm{EFA}}^2 = \frac{[1-(r_\eta)^2]\mathrm{e}^{\mathrm{j}(K_z-k_{1z})H}}{1-(r_\eta)^2\mathrm{e}^{2\mathrm{j}K_zH}}. \tag{81}$$

Moreover, for higher frequencies, the transmission coefficients decrease with increasing (i) the volume concentration, and (ii) the geometrical thickness of the layer. To justify the result $T_\eta^2 \approx T_{\eta\mathrm{EFA}}^2$, we consider Eq. (33) of the generalized Ewald–Oseen extinction theorem, in which we employ the assumption $|x_{-\eta n}^{1,2}| \ll |x_{+\eta n}^{1,2}|$; we then get

$$1 = -\mathrm{j}\frac{8\pi^2 n_0}{k_1k_{1z}}\frac{1}{K_z-k_{1z}}\sum_n \chi_n\sqrt{n(n+1)} \times \left[M_n(\widehat{\mathbf{s}}\cdot\widehat{\mathbf{s}}_{\mathrm{T}})T_n^1x_{+\theta n}^1 - N_n(\widehat{\mathbf{s}}\cdot\widehat{\mathbf{s}}_{\mathrm{T}})T_n^2x_{+\theta n}^2\right], \tag{82}$$

so that from Eqs. (79) and (80), we find

$$T_\eta^2 \approx \mathrm{e}^{\mathrm{j}(K_z-k_{1z})H}. \tag{83}$$

On the other hand, it is obvious from Eq. (81) that the same approximation is valid for $T_{\eta\mathrm{EFA}}^2$ under the assumption $|r_\eta^2| \ll 1$. It should be pointed out that this result was exploited by Gustavsson et al. [11] to compute the effective wavenumber in the framework of the integral equation method developed by Kristensson [12] to analyze the coherent scattering by a discrete random layer.

From the results of Figs. 4 and 5, we deduce that only the reflection data contain information of the particulate structure of the layer (except for low frequencies, when the effective field approximation applies) [11].

3.3. The coherent field inside the layer

In this section we present several methods for computing the coherent field inside the layer.

3.3.1. The method of Fikioris and Waterman

We consider the coherent field $\mathbf{E}_c(\mathbf{r})$ given by Eq. (44). From Eqs. (45)–(47) it is apparent that the computation of $\mathbf{E}_c(\mathbf{r})$ requires the calculation of the integrals

$$\int_{D-D_a(\mathbf{r})} \mathbf{M}_{mn}^3(k_1\mathbf{r}_i)e^{j\mathbf{K}_b\cdot\mathbf{R}_i}d^3\mathbf{R}_i, \quad \int_{D-D_a(\mathbf{r})} \mathbf{N}_{mn}^3(k_1\mathbf{r}_i)e^{j\mathbf{K}_b\cdot\mathbf{R}_i}d^3\mathbf{R}_i \tag{84}$$

and

$$\int_{D_a(\mathbf{r})} \mathbf{M}_{mn}^1(k_{1,2}\mathbf{r}_i)e^{j\mathbf{K}_b\cdot\mathbf{R}_i}d^3\mathbf{R}_i, \quad \int_{D_a(\mathbf{r})} \mathbf{N}_{mn}^1(k_{1,2}\mathbf{r}_i)e^{j\mathbf{K}_b\cdot\mathbf{R}_i}d^3\mathbf{R}_i. \tag{85}$$

These volume integrals are transformed into surface integrals by means of the Green's theorem of the second kind. For the scalar function $f = \exp(j\mathbf{K}_b\cdot\mathbf{R}_i)$ and the vector function $\mathbf{G} = \mathbf{M}_{mn}^{1,3}(k\mathbf{r}_i)$ or $\mathbf{N}_{mn}^{1,3}(k\mathbf{r}_i)$, solving the Helmholtz equation in the domain D_0 with the wavenumbers K and k, respectively, the Green's theorem yields

$$\int_{D_0} f\mathbf{G}\,dV = \frac{1}{K^2-k^2}\int_{S_0}\left(f\frac{\partial\mathbf{G}}{\partial\widehat{\mathbf{n}}} - \mathbf{G}\frac{\partial f}{\partial\widehat{\mathbf{n}}}\right)dS, \tag{86}$$

where S_0 is the boundary surface of D_0, and $\widehat{\mathbf{n}}$ is the outward-pointing normal unit vector to S_0. Actually, the volume integral over $D-D_a(\mathbf{r})$ will be transformed into three surface integrals (we neglect the integrals over the cylindrical surfaces at infinity), i.e.,

$$\int_{D-D_a(\mathbf{r})} dV \to \int_{S_{z0}\cup S_{zH}} \widehat{\mathbf{n}}_z^+ dS - \int_{S_a(\mathbf{r})} \widehat{\mathbf{n}}_a^+ dS, \tag{87}$$

while the volume integral over $D_a(\mathbf{r})$ will be transformed into one surface integral, i.e.,

$$\int_{D_a(\mathbf{r})} dV \to \int_{S_a(\mathbf{r})} \widehat{\mathbf{n}}_a^+ dS. \tag{88}$$

Here, $\widehat{\mathbf{n}}_a^+$ is the outward-pointing unit vector normal to the spherical surface $S_a(\mathbf{r})$ of radius a centered at $\mathbf{r}$, and $\widehat{\mathbf{n}}_z^+ = -\widehat{\mathbf{z}}$ and $\widehat{\mathbf{n}}_z^+ = \widehat{\mathbf{z}}$ are the outward-pointing unit vectors normal to the plane surfaces S_{z0} and S_{zH}, respectively.

Let us denote by

1. $\mathbf{I}_{zMmn}^b(\mathbf{r})$ and $\mathbf{I}_{zNmn}^b(\mathbf{r})$ the integrals over the union of the planes $S_{z0}\cup S_{zH}$ involving $\mathbf{M}_{mn}^3(k_1\mathbf{r}_i)$ and $\mathbf{N}_{mn}^3(k_1\mathbf{r}_i)$, respectively; and by
2. $\mathbf{I}_{aMmn}^{\alpha b}(\mathbf{r})$ and $\mathbf{I}_{aNmn}^{\alpha b}(\mathbf{r})$ the integrals over the spherical surface $S_a(\mathbf{r})$ involving $\mathbf{M}_{mn}^\alpha(k_\alpha\mathbf{r}_i)$ and $\mathbf{N}_{mn}^\alpha(k_\alpha\mathbf{r}_i)$, respectively.

In the latter case, the vector spherical wave functions $\mathbf{M}_{mn}^\alpha(k_\alpha\mathbf{r}_i)$ and $\mathbf{N}_{mn}^\alpha(k_\alpha\mathbf{r}_i)$ are defined as follows:

1. for $\alpha=1$, we have $k_\alpha = k_1$, $\mathbf{M}_{mn}^\alpha = \mathbf{M}_{mn}^1$, and $\mathbf{N}_{mn}^\alpha = \mathbf{N}_{mn}^1$;
2. for $\alpha=2$, we have $k_\alpha = k_2$, $\mathbf{M}_{mn}^\alpha = \mathbf{M}_{mn}^1$, and $\mathbf{N}_{mn}^\alpha = \mathbf{N}_{mn}^1$; and
3. for $\alpha=3$, we have $k_\alpha = k_1$, $\mathbf{M}_{mn}^\alpha = \mathbf{M}_{mn}^3$, and $\mathbf{N}_{mn}^\alpha = \mathbf{N}_{mn}^3$.

With this notation, we express the coherent field as

$$\mathbf{E}_{c\eta}(\mathbf{r}) = \mathbf{E}_{c\eta1}(\mathbf{r}) + \mathbf{E}_{c\eta2}(\mathbf{r}), \tag{89}$$

where $\mathbf{E}_{c\eta1}(\mathbf{r})$ and $\mathbf{E}_{c\eta2}(\mathbf{r})$ are given by

$$\mathbf{E}_{c\eta1}(\mathbf{r}) = \mathbf{E}_{0\eta}(\mathbf{r}) + n_0\sum_{b=\pm}\sum_{mn}\left[T_n^1 e_{b\eta mn}^1\mathbf{I}_{zMmn}^b(\mathbf{r}) + T_n^2 e_{b\eta mn}^2\mathbf{I}_{zNmn}^b(\mathbf{r})\right], \tag{90}$$

and

$$\mathbf{E}_{c\eta2}(\mathbf{r}) = \sum_{\alpha=1}^{3}(-1)^\alpha\mathbf{E}_{c\eta2}^\alpha(\mathbf{r}), \tag{91}$$

$$\mathbf{E}_{c\eta2}^\alpha(\mathbf{r}) = n_0\sum_{b=\pm}\sum_{mn}\left[T_{\alpha n}^1 e_{b\eta mn}^1\mathbf{I}_{aMmn}^{\alpha b}(\mathbf{r}) + T_{\alpha n}^2 e_{b\eta mn}^2\mathbf{I}_{aNmn}^{\alpha b}(\mathbf{r})\right], \tag{92}$$

respectively. In Eq. (92), we used the convention that (i) $T_{\alpha n}^{1,2}=1$ for $\alpha=1$ (exciting field), (ii) $T_{\alpha n}^{1,2} = T_{\mathrm{int}n}^{1,2}$ for $\alpha=2$ (internal field), and (iii) $T_{\alpha n}^{1,2} = T_n^{1,2}$ for $\alpha=3$ (scattered field).

In the following, we will calculate the components $\mathbf{E}_{c\eta1}(\mathbf{r})$ and $\mathbf{E}_{c\eta2}(\mathbf{r})$ of the coherent field separately. In fact, we will show that $\mathbf{E}_{c\eta1}(\mathbf{r}) = 0$, so that $\mathbf{E}_{c\eta}(\mathbf{r}) = \mathbf{E}_{c\eta2}(\mathbf{r})$. In view of Eq. (90), the result $\mathbf{E}_{c\eta1}(\mathbf{r}) = 0$ means that the waves produced by the particles situated at the lower and upper boundaries of the medium extinguishes the incident wave.

3.3.1.1. Computation of $\mathbf{E}_{c\eta1}$.

To compute $\mathbf{I}_{zMmn}^b(\mathbf{r})$ and $\mathbf{I}_{zNmn}^b(\mathbf{r})$, we employ the integral representations of the radiating vector spherical wave functions in terms of plane electromagnetic waves; in particular, for $z\ge a$ and in the plane $p_z=-z$ ($z_i=0$), we use ($\mathbf{p} = \mathbf{R}_i - \mathbf{r} = -\mathbf{r}_i$)

$$\mathbf{M}_{mn}^3(k_1\mathbf{p}) = \frac{1}{2\pi j^n}\int \mathbf{m}_{mn}(\widehat{\mathbf{k}}^-)e^{j[\mathbf{k}_\perp\cdot\mathbf{p}_\perp + k_{1z}(\mathbf{k}_\perp)z]}\frac{d^2\mathbf{k}_\perp}{k_1k_{1z}(\mathbf{k}_\perp)},$$

$$\mathbf{N}_{mn}^3(k_1\mathbf{p}) = \frac{1}{2\pi j^n}\int j\mathbf{n}_{mn}(\widehat{\mathbf{k}}^-)e^{j[\mathbf{k}_\perp\cdot\mathbf{p}_\perp + k_{1z}(\mathbf{k}_\perp)z]}\frac{d^2\mathbf{k}_\perp}{k_1k_{1z}(\mathbf{k}_\perp)},$$

while for $z\le H-a$ and in the plane $p_z = H-z$ ($z_i=H$), we use

$$\mathbf{M}_{mn}^3(k_1\mathbf{p}) = \frac{1}{2\pi j^n}\int \mathbf{m}_{mn}(\widehat{\mathbf{k}}^+)e^{j[\mathbf{k}_\perp\cdot\mathbf{p}_\perp + k_{1z}(\mathbf{k}_\perp)(H-z)]}\frac{d^2\mathbf{k}_\perp}{k_1k_{1z}(\mathbf{k}_\perp)},$$

$$\mathbf{N}_{mn}^3(k_1\mathbf{p}) = \frac{1}{2\pi j^n}\int j\mathbf{n}_{mn}(\widehat{\mathbf{k}}^+)e^{j[\mathbf{k}_\perp\cdot\mathbf{p}_\perp + k_{1z}(\mathbf{k}_\perp)(H-z)]}\frac{d^2\mathbf{k}_\perp}{k_1k_{1z}(\mathbf{k}_\perp)},$$

where $\mathbf{k}^b = \mathbf{k}_\perp + bk_{1z}(\mathbf{k}_\perp)\widehat{\mathbf{z}}$ for $b=\pm$ and $k_{1z}(\mathbf{k}_\perp) = \sqrt{k_1^2 - p_\perp^2}$. We obtain

$$\mathbf{I}_{zMmn}^b(\mathbf{r}) = -\frac{2\pi}{j^{n+1}k_1k_{1z}(bK_z-k_{1z})}e^{j\mathbf{k}_{1s}\cdot\mathbf{r}}\mathbf{m}_{mn}(\widehat{\mathbf{s}}) + \frac{2\pi}{j^{n+1}k_1k_{1z}(bK_z+k_{1z})}e^{j\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp}e^{jk_{1z}(H-z)}e^{jbK_zH}\mathbf{m}_{mn}(\widehat{\mathbf{s}}_R) \tag{93}$$

and

$$\mathbf{I}_{zNmn}^b(\mathbf{r}) = -\frac{2\pi j}{j^{n+1}k_1k_{1z}(bK_z-k_{1z})}e^{j\mathbf{k}_{1s}\cdot\mathbf{r}}\mathbf{n}_{mn}(\widehat{\mathbf{s}}) + \frac{2\pi j}{j^{n+1}k_1k_{1z}(bK_z+k_{1z})}e^{j\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp}e^{jk_{1z}(H-z)}e^{jbK_zH}\mathbf{n}_{mn}(\widehat{\mathbf{s}}_R), \tag{94}$$

so that the field $\mathbf{E}_{c\eta1}(\mathbf{r})$ computes as

$$\begin{aligned}\mathbf{E}_{c\eta1}(\mathbf{r}) = {}& \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}})e^{j\mathbf{k}_{1s}\cdot\mathbf{r}} - \frac{2\pi n_0}{k_1k_{1z}}e^{j\mathbf{k}_{1s}\cdot\mathbf{r}}\sum_{b=\pm}\frac{1}{bK_z-k_{1z}} \\ &\times\sum_{mn}\frac{1}{j^{n+1}}\left[T_n^1e_{b\eta mn}^1\mathbf{m}_{mn}(\widehat{\mathbf{s}}) + jT_n^2e_{b\eta mn}^2\mathbf{n}_{mn}(\widehat{\mathbf{s}})\right] \\ &+ \frac{2\pi n_0}{k_1k_{1z}}e^{j\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp}e^{jk_{1z}(H-z)}\sum_{b=\pm}\frac{e^{jbK_zH}}{bK_z+k_{1z}} \\ &\times\sum_{mn}\frac{1}{j^{n+1}}\left[T_n^1e_{b\eta mn}^1\mathbf{m}_{mn}(\widehat{\mathbf{s}}_R) + jT_n^2e_{b\eta mn}^2\mathbf{n}_{mn}(\widehat{\mathbf{s}}_R)\right].\end{aligned} \tag{95}$$

Next we proceed as follows.

1. Let $\mathbf{E}^1_{c\eta 1}(\mathbf{r})$ be the cumulative contribution of the first two terms on the right-hand side of Eq. (95). By means of Eqs. (24) and (25), this contribution can be written as

$$\mathbf{E}^1_{c\eta 1}(\mathbf{r}) = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}})\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s}\cdot\mathbf{r}} + \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s}\cdot\mathbf{r}}\sum_{b=\pm}\frac{1}{bK_z - k_{1z}} \times \sum_{mn}\{[\mathbf{m}_{mn}(\widehat{\mathbf{s}})\otimes\mathbf{m}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T^1_n x^1_{b\eta n} + [\mathbf{n}_{mn}(\widehat{\mathbf{s}})\otimes\mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T^2_n x^2_{b\eta n}\}. \quad (96)$$

Using Eqs. (26)–(29) along with Eqs. (33)–(35) of the generalized Ewald–Oseen extinction theorem, we find that $\mathbf{E}^1_{c\eta 1}(\mathbf{r}) = 0$.

2. Let $\mathbf{E}^2_{c\eta 1}(\mathbf{r})$ be the contribution of the third term on the right-hand side of Eq. (95). Again, by means of Eqs. (24) and (25), we have

$$\mathbf{E}^2_{c\eta 1}(\mathbf{r}) = -\mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp}\mathrm{e}^{\mathrm{j}k_{1z}(H-z)}\sum_{b=\pm}\frac{\mathrm{e}^{\mathrm{j}bK_z H}}{bK_z + k_{1z}} \times \sum_{mn}\Big\{[\mathbf{m}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})\otimes\mathbf{m}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T^1_n x^1_{b\eta n} + [\mathbf{n}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})\otimes\mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T^2_n x^2_{b\eta n}\Big\}. \quad (97)$$

Using Eqs. (26)–(29) along with Eqs. (34)–(36) of the generalized Ewald–Oseen extinction theorem, we find that $\mathbf{E}^2_{c\eta 1}(\mathbf{r}) = 0$.

Thus, the field $\mathbf{E}_{c\eta 1}(\mathbf{r}) = \mathbf{E}^1_{c\eta 1}(\mathbf{r}) + \mathbf{E}^2_{c\eta 1}(\mathbf{r})$ vanishes, and we have $\mathbf{E}_{c\eta}(\mathbf{r}) = \mathbf{E}_{c\eta 2}(\mathbf{r})$.

3.3.1.2. Computation of $\mathbf{E}_{c\eta 2}$. To compute $\mathbf{I}^{\alpha b}_{a\mathrm{M}mn}(\mathbf{r})$ and $\mathbf{I}^{\alpha b}_{a\mathrm{N}mn}(\mathbf{r})$, we make the change of variable $\mathbf{p} = \mathbf{R}_i - \mathbf{r} = -\mathbf{r}_i$, and use (i) the expansion (cf. Appendix A)

$$\bar{\mathbf{I}}\mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{p}} = \bar{\mathbf{I}}\mathrm{e}^{\mathrm{j}K\widehat{\mathbf{s}}_b\cdot\mathbf{p}} = -4\pi\sum_{m'n'}\mathrm{j}^{n'+1}[\mathbf{l}_{-m'n'}(\widehat{\mathbf{s}}_b)\otimes\mathbf{L}^1_{m'n'}(K\mathbf{p}) + \mathrm{j}\mathbf{m}_{-m'n'}(\widehat{\mathbf{s}}_b)\otimes\mathbf{M}^1_{m'n'}(K\mathbf{p}) + \mathbf{n}_{-m'n'}(\widehat{\mathbf{s}}_b)\otimes\mathbf{N}^1_{m'n'}(K\mathbf{p})]; \quad (98)$$

(ii) the dyadic identities $\mathbf{a} = \bar{\mathbf{I}}\cdot\mathbf{a}$ and $(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$; (iii) the relations

$$\int\Big[k_\alpha\mathbf{M}^1_{m'n'}(K\mathbf{p})\cdot\frac{\partial}{\partial(k_\alpha p)}\mathbf{M}^\alpha_{mn}(k_\alpha\mathbf{p}) - K\mathbf{M}^\alpha_{mn}(k_\alpha\mathbf{p})\cdot\frac{\partial}{\partial(Kp)}\mathbf{M}^1_{m'n'}(K\mathbf{p})\Big]\mathrm{d}^2\widehat{\mathbf{p}} = \int\Big\{k_\alpha\widehat{\mathbf{p}}\cdot[\mathbf{M}^1_{m'n'}(K\mathbf{p})\times\mathbf{N}^\alpha_{mn}(k_\alpha\mathbf{p})] + K\widehat{\mathbf{p}}\cdot[\mathbf{N}^1_{m'n'}(K\mathbf{p})\times\mathbf{M}^\alpha_{mn}(k_\alpha\mathbf{p})]\Big\}\mathrm{d}^2\widehat{\mathbf{p}}, \quad (99)$$

$$\int\Big[k_\alpha\mathbf{N}^1_{m'n'}(K\mathbf{p})\cdot\frac{\partial}{\partial(k_\alpha p)}\mathbf{N}^\alpha_{mn}(k_\alpha\mathbf{p}) - K\mathbf{N}^\alpha_{mn}(k_\alpha\mathbf{p})\cdot\frac{\partial}{\partial(Kp)}\mathbf{N}^1_{m'n'}(K\mathbf{p})\Big]\mathrm{d}^2\widehat{\mathbf{p}} = \int\Big\{k_\alpha\widehat{\mathbf{p}}\cdot[\mathbf{N}^1_{m'n'}(K\mathbf{p})\times\mathbf{M}^\alpha_{mn}(k_\alpha\mathbf{p})] + K\widehat{\mathbf{p}}\cdot[\mathbf{M}^1_{m'n'}(K\mathbf{p})\times\mathbf{N}^\alpha_{mn}(k_\alpha\mathbf{p})]\Big\}\mathrm{d}^2\widehat{\mathbf{p}}, \quad (100)$$

and

$$\frac{1}{K^2-k_\alpha^2}\int\Big[k_\alpha\mathbf{L}^1_{m'n'}(K\mathbf{p})\cdot\frac{\partial}{\partial(k_\alpha p)}\mathbf{N}^\alpha_{mn}(k_\alpha\mathbf{p}) - K\mathbf{N}^\alpha_{mn}(k_\alpha\mathbf{p})\cdot\frac{\partial}{\partial(Kp)}\mathbf{L}^1_{m'n'}(K\mathbf{p})\Big]\mathrm{d}^2\widehat{\mathbf{p}} = -\frac{1}{k_\alpha}\int\mathbf{M}^\alpha_{mn}(k_\alpha\mathbf{p})\cdot[\widehat{\mathbf{p}}\times\mathbf{L}^1_{m'n'}(K\mathbf{p})]\mathrm{d}^2\widehat{\mathbf{p}}; \quad (101)$$

(iv) the representation of the vector spherical wave functions in terms of the vector spherical harmonics $\mathbf{v}^1_{mn}(\widehat{\mathbf{p}}) = \mathbf{l}_{mn}(\widehat{\mathbf{p}})$, $\mathbf{v}^2_{mn}(\widehat{\mathbf{p}}) = \mathbf{n}_{mn}(\widehat{\mathbf{p}})$, and $\mathbf{v}^3_{mn}(\widehat{\mathbf{p}}) = \mathbf{m}_{mn}(\widehat{\mathbf{p}})$; (v) the orthogonality relations (cf. Appendix A of Ref. [1])

$$\int\mathbf{v}^\alpha_{mn}(\widehat{\mathbf{p}})\cdot\mathbf{v}^\beta_{m'n'}(\widehat{\mathbf{p}})\,\mathrm{d}^2\widehat{\mathbf{p}} = \delta_{pq}\delta_{m',-m}\delta_{n'n},\quad \alpha,\beta = 1,2,3; \quad (102)$$

and finally, (vi) the identities

$$\widehat{\mathbf{p}}\times\mathbf{l}_{mn}(\widehat{\mathbf{p}}) = 0,\ \widehat{\mathbf{p}}\times\mathbf{n}_{mn}(\widehat{\mathbf{p}}) = -\mathbf{m}_{mn}(\widehat{\mathbf{p}}),\ \widehat{\mathbf{p}}\times\mathbf{m}_{mn}(\widehat{\mathbf{p}}) = \mathbf{n}_{mn}(\widehat{\mathbf{p}}). \quad (103)$$

Note that the relations (99)–(101) have been derived by using the representations of the vector spherical wave functions in terms of vector spherical harmonics (cf. Appendix A of Ref. [1]) and the differential equation satisfied by the spherical Bessel and Hankel functions $z_n(x)$, i.e.,

$$x^2 z_n''(x) + 2xz_n'(x) + [x^2 - n(n+1)]z_n(x) = 0. \quad (104)$$

We obtain

$$\mathbf{I}^{\alpha b}_{a\mathrm{M}mn}(\mathbf{r}) = 4\pi\mathrm{j}^n I^{\mathrm{M}}_{n\alpha}(a)\mathbf{m}_{mn}(\widehat{\mathbf{s}}_b), \quad (105)$$

$$\mathbf{I}^{\alpha b}_{a\mathrm{N}mn}(\mathbf{r}) = -4\pi\mathrm{j}^{n+1}[I^{\mathrm{L}}_{n\alpha}(a)\mathbf{l}_{mn}(\widehat{\mathbf{s}}_b) + I^{\mathrm{N}}_{n\alpha}(a)\mathbf{n}_{mn}(\widehat{\mathbf{s}}_b)], \quad (106)$$

where

$$I^{\mathrm{M}}_{n\alpha}(a) = \frac{a}{K^2-k_\alpha^2}\Big\{j_n(Ka)[k_\alpha a z^\alpha_n(k_\alpha a)]' - z^\alpha_n(k_\alpha a)[Kaj_n(Ka)]'\Big\}, \quad (107)$$

$$I^{\mathrm{N}}_{n\alpha}(a) = \frac{a}{K^2-k_\alpha^2}\Big\{\frac{K}{k_\alpha}j_n(Ka)[k_\alpha a z^\alpha_n(k_\alpha a)]' - \frac{k_\alpha}{K}z^\alpha_n(k_\alpha a)[Kaj_n(Ka)]'\Big\}, \quad (108)$$

$$I^{\mathrm{L}}_{n\alpha}(a) = \frac{a}{Kk_\alpha}\sqrt{n(n+1)}z^\alpha_n(k_\alpha a)j_n(Ka), \quad (109)$$

$z^1_n(x) = z^2_n(x) = j_n(x)$, $z^3_n(x) = h_n(x)$, and $j_n(x)$ and $h_n(x)$ are the spherical Bessel and Hankel functions, respectively. Thus, the fields $\mathbf{E}^\alpha_{c\eta 2}(\mathbf{r})$ result from Eq. (92) along with Eqs. (105)–(109).

Before continuing our analysis, we make a short comment. In view of Eq. (98), the volume integrals

$$\int_{D_a}\mathbf{M}^1_{mn}(k_\alpha\mathbf{p})\mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{p}}\,\mathrm{d}^3\mathbf{p}\ \text{and}\ \int_{D_a}\mathbf{N}^1_{mn}(k_\alpha\mathbf{p})\mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{p}}\,\mathrm{d}^3\mathbf{p}, \quad (110)$$

where $\alpha = 1, 2$ and D_a is the domain occupied by a spherical particle of radius a centered at the origin of the coordinate system, reduces to the calculation of the following integrals:

$$\int_{D_a}\mathbf{M}^1_{m'n'}(K\mathbf{p})\cdot\begin{pmatrix}\mathbf{M}^1_{mn}(k_\alpha\mathbf{p})\\ \mathbf{N}^1_{mn}(k_\alpha\mathbf{p})\end{pmatrix}\mathrm{d}^3\mathbf{p}, \quad (111)$$

$$\int_{D_a}\mathbf{N}^1_{m'n'}(K\mathbf{p})\cdot\begin{pmatrix}\mathbf{M}^1_{mn}(k_\alpha\mathbf{p})\\ \mathbf{N}^1_{mn}(k_\alpha\mathbf{p})\end{pmatrix}\mathrm{d}^3\mathbf{p}, \quad (112)$$

and

$$\int_{D_a}\mathbf{L}^1_{m'n'}(K\mathbf{p})\cdot\begin{pmatrix}\mathbf{M}^1_{mn}(k_\alpha\mathbf{p})\\ \mathbf{N}^1_{mn}(k_\alpha\mathbf{p})\end{pmatrix}\mathrm{d}^3\mathbf{p}. \quad (113)$$

Instead of using the Green's theorem (86), we can compute the integrals (111)–(112) by means of the second vector Green's theorem.

For $\mathbf{P}(\mathbf{r}) = \mathbf{M}^1_{m'n'}(K\mathbf{r})/\mathbf{N}^1_{m'n'}(K\mathbf{r})$, $\mathbf{Q}(\mathbf{r}) = \mathbf{M}^1_{mn}(k_\alpha\mathbf{r})/\mathbf{N}^1_{mn}(k_\alpha\mathbf{r})$, and a spherical domain, we use

$$\int_{D_a} [\mathbf{P}\cdot(\nabla\times\nabla\times\mathbf{Q}) - \mathbf{Q}\cdot(\nabla\times\nabla\times\mathbf{P})]\,\mathrm{d}^3\mathbf{r}$$
$$= a^2\int \widehat{\mathbf{r}}\cdot[\mathbf{Q}\times(\nabla\times\mathbf{P}) - \mathbf{P}\times(\nabla\times\mathbf{Q})]\,\mathrm{d}^2\widehat{\mathbf{r}}. \tag{114}$$

On the other hand, by taking into account that $\nabla\times\mathbf{L}^1_{m'n'} = 0$, the integral (113) can be computed by means of the first vector Green's theorem. For $\mathbf{P}(\mathbf{r}) = \mathbf{L}^1_{m'n'}(K\mathbf{r})$, $\mathbf{Q}(\mathbf{r}) = \mathbf{M}^1_{mn}(k_\alpha\mathbf{r})/\mathbf{N}^1_{mn}(k_\alpha\mathbf{r})$, and a spherical domain, we use

$$\int_{D_a} [(\nabla\times\mathbf{P})\cdot(\nabla\times\mathbf{Q}) - \mathbf{P}\cdot(\nabla\times\nabla\times\mathbf{Q})]\,\mathrm{d}^3\mathbf{r}$$
$$= a^2\int \widehat{\mathbf{r}}\cdot[\mathbf{P}\times(\nabla\times\mathbf{Q})]\,\mathrm{d}^2\widehat{\mathbf{r}}. \tag{115}$$

The final expressions for $\mathbf{I}^{\alpha b}_{a\mathrm{M}mn}(\mathbf{r})$ and $\mathbf{I}^{\alpha b}_{a\mathrm{N}mn}(\mathbf{r})$ are as in Eqs. (105) and (106) along with Eqs. (107)–(109); however, the proof is now more straightforward and does not make use on the identities (99)–(101).

3.3.1.3. Computation of $\mathbf{E}_{c\eta}$. We are now in a position to obtain a complete representation for the coherent field. Inserting Eqs. (105) and (106) together with Eqs. (24) and (25) in Eq. (92), and the result in Eq. (91), we get

$$\mathbf{E}_{c\eta}(\mathbf{r}) = 16\pi^2 n_0 \sum_{b=\pm} \mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{r}}$$
$$\times\sum_{mn}\Bigg\{[\mathbf{m}_{mn}(\widehat{\mathbf{s}}_b)\otimes\mathbf{m}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)\Bigg[\sum_{\alpha=1}^{3}(-1)^\alpha T^1_{\alpha n} I^{\mathrm{M}}_{n\alpha}(a)\Bigg]x^1_{b\eta n}$$
$$+[\mathbf{n}_{mn}(\widehat{\mathbf{s}}_b)\otimes\mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)\Bigg[\sum_{\alpha=1}^{3}(-1)^\alpha T^2_{\alpha n} I^{\mathrm{N}}_{n\alpha}(a)\Bigg]x^2_{b\eta n}$$
$$+[\mathbf{l}_{mn}(\widehat{\mathbf{s}}_b)\otimes\mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)\Bigg[\sum_{\alpha=1}^{3}(-1)^\alpha T^2_{\alpha n} I^{\mathrm{L}}_{n\alpha}(a)\Bigg]x^2_{b\eta n}\Bigg\}. \tag{116}$$

Next, we use the following result. The addition theorem for the vector spherical harmonics $\mathbf{l}_{mn}(\widehat{\mathbf{k}})$ and $\mathbf{n}_{-mn}(\widehat{\mathbf{k}}')$ in the case $\varphi=\varphi'$ is (cf. Appendix B of Ref. [1])

$$\sum_m \mathbf{l}_{mn}(\widehat{\mathbf{k}})\otimes\mathbf{n}_{-mn}(\widehat{\mathbf{k}}') = \chi_n\frac{\partial P_n(x)}{\partial\theta'}\widehat{\mathbf{k}}\otimes\widehat{\boldsymbol{\theta}}(\widehat{\mathbf{k}}'), \tag{117}$$

where $\widehat{\mathbf{k}} = \widehat{\mathbf{k}}(\theta,\varphi)$, $\widehat{\mathbf{k}}' = \widehat{\mathbf{k}}'(\theta',\varphi)$, $x = \widehat{\mathbf{k}}\cdot\widehat{\mathbf{k}}' = \cos(\theta-\theta')$,

$$\frac{\partial P_n(x)}{\partial\theta'} = \sqrt{n(n+1)}L_n(\widehat{\mathbf{k}},\widehat{\mathbf{k}}'), \tag{118}$$

$$L_n(\widehat{\mathbf{k}},\widehat{\mathbf{k}}') = \sin(\theta-\theta')\pi^1_n(x), \tag{119}$$

and $P_n(x) = P^0_n(x)$ are the Legendre polynomials. Since for $\widehat{\mathbf{k}} = \widehat{\mathbf{k}}'$ we have $L_n(\widehat{\mathbf{k}},\widehat{\mathbf{k}}) = 0$, it is readily seen that

$$\sum_m [\mathbf{l}_{mn}(\widehat{\mathbf{s}}_b)\otimes\mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b) = 0. \tag{120}$$

Thus, the last term in Eq. (116) vanishes. Substituting Eqs. (26)–(29) in Eq. (116), using the special values of the functions M_n and N_n at $x=1$ (cf. Eqs. (31) and (32)),

$$M_n(1) = -N_n(1) = \omega_n, \tag{121}$$

with

$$\omega_n = \frac{1}{2}\sqrt{\frac{n(n+1)(2n+1)}{2}}, \tag{122}$$

we find that the coherent field can be written in the form

$$\mathbf{E}_{c\eta}(\mathbf{r}) = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{T}})T^1_{\eta\mathrm{FW}}\mathrm{e}^{\mathrm{j}K\widehat{\mathbf{s}}_{\mathrm{T}}\cdot\mathbf{r}} + \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{TR}})R^1_{\eta\mathrm{FW}}\mathrm{e}^{\mathrm{j}K\widehat{\mathbf{s}}_{\mathrm{TR}}\cdot\mathbf{r}}$$
$$= \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{T}})T^1_{\eta\mathrm{FW}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}\perp}\cdot\mathbf{r}}\mathrm{e}^{\mathrm{j}K_z z} + \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{TR}})R^1_{\eta\mathrm{FW}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}\perp}\cdot\mathbf{r}}\mathrm{e}^{-\mathrm{j}K_z z}, \tag{123}$$

where the transmission and reflection coefficients are given by

$$T^1_{\eta\mathrm{FW}} = 2\pi n_0\sum_{mn}(2n+1)\Bigg\{\Bigg[\sum_{\alpha=1}^{3}(-1)^\alpha T^1_{\alpha n}I^{\mathrm{M}}_{n\alpha}(a)\Bigg]x^1_{+\eta n}$$
$$+\Bigg[\sum_{\alpha=1}^{3}(-1)^\alpha T^2_{\alpha n}I^{\mathrm{N}}_{n\alpha}(a)\Bigg]x^2_{+\eta n}\Bigg\} \tag{124}$$

and

$$R^1_{\eta\mathrm{FW}} = 2\pi n_0\sum_{mn}(2n+1)\Bigg\{\Bigg[\sum_{\alpha=1}^{3}(-1)^\alpha T^1_{\alpha n}I^{\mathrm{M}}_{n\alpha}(a)\Bigg]x^1_{-\eta n}$$
$$+\Bigg[\sum_{\alpha=1}^{3}(-1)^\alpha T^2_{\alpha n}I^{\mathrm{N}}_{n\alpha}(a)\Bigg]x^2_{-\eta n}\Bigg\}, \tag{125}$$

respectively.

From Eq. (123), we see that the coherent field $\mathbf{E}_{c\eta}(\mathbf{r})$ is a superposition of upwelling and downwelling plane electromagnetic waves propagating in an effective medium with wavenumber K. If we extend the representations of the coherent fields reflected and transmitted by the layer in the critical domains $-a<z\le 0$ and $H\le z<H+a$, respectively, as well as the representation of the coherent field in the critical domains $0\le z<a$ and $H-a<z\le H$, we may expect that the boundary conditions for the electric fields,

$$\widehat{\mathbf{z}}\times\mathbf{E}_{c\eta}(\mathbf{r}) = \widehat{\mathbf{z}}\times[\mathbf{E}_{\mathrm{R}\eta}(\mathbf{r}) + \mathbf{E}_{0\eta}(\mathbf{r})] \text{ at } z=0 \tag{126}$$

and

$$\widehat{\mathbf{z}}\times\mathbf{E}_{c\eta}(\mathbf{r}) = \widehat{\mathbf{z}}\times\mathbf{E}_{\mathrm{T}\eta}(\mathbf{r}) \text{ at } z=H, \tag{127}$$

are satisfied. Fikioris and Waterman [3,4] proved that the zeroth-order fields satisfy the boundary conditions (126) and (127) in the low frequency limit $k_1a\ll 1$. Unfortunately, at higher frequencies, these boundary conditions are not satisfied. This can be inferred from Fig. 6 illustrating the relative errors in the reflection and transmission coefficients

$$\varepsilon_{\mathrm{R}\theta} = \frac{||R^1_\theta| - |R^1_{\theta\mathrm{FW}}||}{|R^1_\theta|},\quad \varepsilon_{\mathrm{T}\theta} = \frac{||T^1_\theta| - |T^1_{\theta\mathrm{FW}}||}{|T^1_\theta|},$$

where $|R^1_\theta|$ and $|T^1_\theta|$ correspond to a model in which the boundary conditions for the electric fields are matched (cf. Eqs. (129) and (130) below). Therefore, for particles with sizes comparable to or larger than the wavelength, higher-order approximations to the fields should by computed by means of the iteration scheme described in Ref. [1] and the validity of the boundary conditions (126) and (127) for each order of approximation should be analyzed [2–4].

3.3.2. The method of Tsang and Kong

The method of Tsang and Kong [6] is a simplified approach that avoids an explicit computation of the coherent field inside the layer, and so, of higher-order approximations. In this approach,

1. the representations of the (zeroth-order) coherent fields reflected and transmitted by the layer are extended in the critical domains;
2. the coherent field inside the layer is assumed to be a superposition of plane electromagnetic waves propagating in an effective medium with wavenumber K, that is,

$$\mathbf{E}_{c\eta}(\mathbf{r}) = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{T}})T^1_\eta\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}\perp}\cdot\mathbf{r}}\mathrm{e}^{\mathrm{j}K_z z} + \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{TR}})R^1_\eta\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}\perp}\cdot\mathbf{r}}\mathrm{e}^{-\mathrm{j}K_z z}; \tag{128}$$

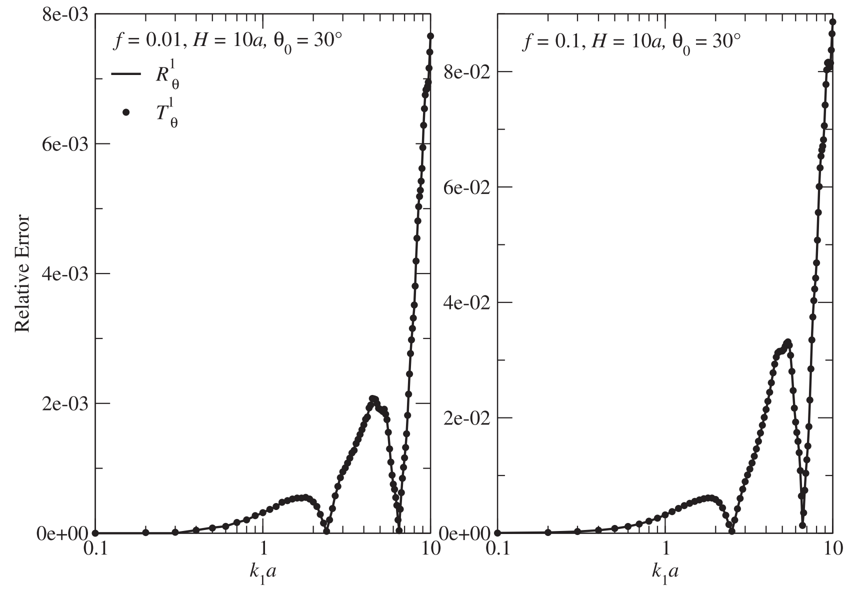

Fig. 6. Relative errors in $|R^1_\theta|$ and $|T^1_\theta|$ for a θ-polarized incidence as functions of the size parameter k_1a and for two values of the volume concentration f. The layer thickness is $H = 10a$, the incidence angle is $\theta_0 = 30^\circ$, the wavenumber of the background medium is $k_1 = 10\,\mu\mathrm{m}^{-1}$, the relative refractive index of the particles is $\mathrm{m} = 1.33$, and the maximum expansion order is $N_{\mathtt{rank}} = 15$. Note that in the case $H = 100a$, the relative errors have virtually the same dependency on k_1a.

3. the reflection and transmission coefficients R^1_η and T^1_η are computed from the boundary conditions for the electric fields (126) and (127).

We obtain

$$R^1_\theta = \frac{1}{1-\mathrm{e}^{2\mathrm{j}K_zH}}\left[(1-R^0_\theta)\mathrm{e}^{2\mathrm{j}K_zH} - T^2_\theta\mathrm{e}^{\mathrm{j}(K_z+k_{1z})H}\right]\frac{\cos\theta_0}{\cos\theta_{\mathrm{T}}}, \tag{129}$$

$$T^1_\theta = (1-R^0_\theta)\frac{\cos\theta_0}{\cos\theta_{\mathrm{T}}} + R^1_\theta \tag{130}$$

for a θ-polarized incidence, and

$$T^1_\varphi = \frac{1}{1-\mathrm{e}^{2\mathrm{j}K_zH}}\left[1+R^0_\varphi - T^2_\varphi\mathrm{e}^{\mathrm{j}(K_z+k_{1z})H}\right], \tag{131}$$

$$R^1_\varphi = 1 + R^0_\varphi - T^1_\varphi \tag{132}$$

for a φ-polarized incidence.

We emphasize once again that the coefficients R^1_η and T^1_η are computed from the boundary conditions for the electric fields, while in the effective field approximation, the coefficients $R^1_{\eta\mathtt{EFA}}$ and $T^1_{\eta\mathtt{EFA}}$, given by

$$R^1_{\theta\mathtt{EFA}} = -\frac{k_1}{K}\frac{r_\eta(1+r_\eta)\mathrm{e}^{2\mathrm{j}K_zH}}{1-(r_\eta)^2\mathrm{e}^{2\mathrm{j}K_zH}},\quad T^1_{\theta\mathtt{EFA}} = \frac{k_1}{K}\frac{1+r_\eta}{1-(r_\eta)^2\mathrm{e}^{2\mathrm{j}K_zH}}, \tag{133}$$

and

$$R^1_{\varphi\mathtt{EFA}} = -\frac{r_\eta(1+r_\eta)\mathrm{e}^{2\mathrm{j}K_zH}}{1-(r_\eta)^2\mathrm{e}^{2\mathrm{j}K_zH}},\quad T^1_{\varphi\mathtt{EFA}} = \frac{1+r_\eta}{1-(r_\eta)^2\mathrm{e}^{2\mathrm{j}K_zH}}, \tag{134}$$

are computed from the boundary conditions for both the electric and magnetic fields. For the magnetic field, the discontinuity in the coefficients R^1_η and T^1_η can be used to predict the effective magnetic permeability of the discrete random layer [4].

3.3.3. Sparse-medium approximation

Let us compute the coherent field $\mathbf{E}_{\mathrm{c}}(\mathbf{r})$ from Eq. (44) by neglecting the integrals over $D_a(\mathbf{r})$. In other words, in the course of evaluating the integrals in Eq. (44) we treat the particles as point scatterers. Note in this respect that the approximation

$$\int_{D-D_a(\mathbf{r})}\mathrm{d}^3\mathbf{R}_i \approx \int_D \mathrm{d}^3\mathbf{R}_i \tag{135}$$

is typical of sparse media and is referred to as the sparse-medium approximation for the integration domain. Also note that in this case, the standard definition of the single particle probability density function $p(\mathbf{R}_i) = 1/V$ is maintained.

The integrals of the radiating vector spherical wave functions are computed in the sense of Cauchy's principal value by excluding from the integration domain D a layer of thickness 2ε around the plane z, and then by letting $\varepsilon \to 0$. We obtain

$$\begin{aligned}\int_D \mathbf{M}^3_{mn}(k_1\mathbf{r}_i)\mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{R}_i}\mathrm{d}^3\mathbf{R}_i &= -\frac{2\pi}{\mathrm{j}^{n+1}k_1k_{1z}(bK_z-k_{1z})}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}}\cdot\mathbf{r}}\mathbf{m}_{mn}(\widehat{\mathbf{s}})\\ &+ \frac{2\pi}{\mathrm{j}^{n+1}k_1k_{1z}(bK_z+k_{1z})}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}\perp}\cdot\mathbf{r}_\perp}\mathrm{e}^{\mathrm{j}k_{1z}(H-z)}\mathrm{e}^{\mathrm{j}bK_zH}\mathbf{m}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})\\ &+ \frac{2\pi}{\mathrm{j}^{n+1}k_1k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}\perp}\cdot\mathbf{r}_\perp}\mathrm{e}^{\mathrm{j}bK_zz}\left[\frac{\mathbf{m}_{mn}(\widehat{\mathbf{s}})}{bK_z-k_{1z}} - \frac{\mathbf{m}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})}{bK_z+k_{1z}}\right]\end{aligned} \tag{136}$$

and

$$\begin{aligned}&\int_D \mathbf{N}^3_{mn}(k_1\mathbf{r}_i)\mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{R}_i}\mathrm{d}^3\mathbf{R}_i\\ &= -\frac{2\pi\mathrm{j}}{\mathrm{j}^{n+1}k_1k_{1z}(bK_z-k_{1z})}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}}\cdot\mathbf{r}}\mathbf{n}_{mn}(\widehat{\mathbf{s}})\\ &+ \frac{2\pi\mathrm{j}}{\mathrm{j}^{n+1}k_1k_{1z}(bK_z+k_{1z})}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}\perp}\cdot\mathbf{r}_\perp}\mathrm{e}^{\mathrm{j}k_{1z}(H-z)}\mathrm{e}^{\mathrm{j}bK_zH}\mathbf{n}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})\\ &+ \frac{2\pi\mathrm{j}}{\mathrm{j}^{n+1}k_1k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{s}\perp}\cdot\mathbf{r}_\perp}\mathrm{e}^{\mathrm{j}bK_zz}\left[\frac{\mathbf{n}_{mn}(\widehat{\mathbf{s}})}{bK_z-k_{1z}} - \frac{\mathbf{n}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})}{bK_z+k_{1z}}\right].\end{aligned} \tag{137}$$

Consequently, we find that the coherent field $\mathbf{E}_{\mathrm{c}\eta}(\mathbf{r})$ is given by

$$\begin{aligned}\mathbf{E}_{c\eta}(\mathbf{r}) &= \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}})\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s}\cdot\mathbf{r}} - \frac{2\pi n_0}{k_1 k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s}\cdot\mathbf{r}}\sum_{b=\pm}\frac{1}{bK_z - k_{1z}} \\ &\times \sum_{mn}\frac{1}{\mathrm{j}^{n+1}}[T_n^1 e^1_{b\eta mn}\mathbf{m}_{mn}(\widehat{\mathbf{s}}) + \mathrm{j}T_n^2 e^2_{b\eta mn}\mathbf{n}_{mn}(\widehat{\mathbf{s}})] \\ &+ \frac{2\pi n_0}{k_1 k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp}\mathrm{e}^{\mathrm{j}k_{1z}(H-z)}\sum_{b=\pm}\frac{\mathrm{e}^{\mathrm{j}bK_zH}}{bK_z + k_{1z}} \\ &\times \sum_{mn}\frac{1}{\mathrm{j}^{n+1}}\left[T_n^1 e^1_{b\eta mn}\mathbf{m}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}}) + \mathrm{j}T_n^2 e^2_{b\eta mn}\mathbf{n}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})\right] \\ &+ \frac{2\pi n_0}{k_1 k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp}\sum_{b=\pm}\mathrm{e}^{\mathrm{j}bK_z z} \\ &\times \sum_{mn}\frac{1}{\mathrm{j}^{n+1}}\left\{T_n^1 e^1_{b\eta mn}\left[\frac{\mathbf{m}_{mn}(\widehat{\mathbf{s}})}{bK_z - k_{1z}} - \frac{\mathbf{m}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})}{bK_z + k_{1z}}\right]\right. \\ &\left. + \mathrm{j}T_n^2 e^2_{b\eta mn}\left[\frac{\mathbf{n}_{mn}(\widehat{\mathbf{s}})}{bK_z - k_{1z}} - \frac{\mathbf{n}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})}{bK_z + k_{1z}}\right]\right\}. \end{aligned} \tag{138}$$

As before, the field $\mathbf{E}^1_{c\eta 1}(\mathbf{r})$ summing the contributions of the first two terms on the right-hand side of Eq. (138), as well as the field $\mathbf{E}^2_{c\eta 1}(\mathbf{r})$ given by the third term on the right-hand side of Eq. (138) vanish. Hence, by taking into account the representations (24) and (25) for the coefficients $e^1_{b\eta mn}$ and $e^2_{b\eta mn}$, respectively, we obtain

$$\begin{aligned}\mathbf{E}_{c\eta}(\mathbf{r}) = &-\mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}}\sum_{b=\pm}\frac{\mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{r}}}{bK_z - k_{1z}} \\ &\times \sum_{mn}\Big\{[\mathbf{m}_{mn}(\widehat{\mathbf{s}}) \otimes \mathbf{m}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T_n^1 x^1_{b\eta n} \\ &+ [\mathbf{n}_{mn}(\widehat{\mathbf{s}}) \otimes \mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T_n^2 x^2_{b\eta n}\Big\} \\ &+ \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}}\sum_{b=\pm}\frac{\mathrm{e}^{\mathrm{j}\mathbf{K}_b\cdot\mathbf{r}}}{bK_z + k_{1z}} \\ &\times \sum_{mn}\Big\{[\mathbf{m}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}}) \otimes \mathbf{m}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T_n^1 x^1_{b\eta n} \\ &+ [\mathbf{n}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}}) \otimes \mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T_n^2 x^2_{b\eta n}\Big\}. \end{aligned} \tag{139}$$

We are now concerned with the computation of the field $\mathbf{E}_{c\eta}(\mathbf{r})$ at the lower and upper boundaries, that is, at $z = 0$ and $z = H$. For this purpose, we extend the representations (57) and (73) for the coherent fields reflected and transmitted by the layer in the critical domains.

1. Case $z = 0$. Using the expression of the coherent reflected field $\mathbf{E}_{\mathrm{R}\eta}(\mathbf{r})$ as given by Eq. (57) in conjunction with Eq. (58), and the result $\mathbf{E}^2_{c\eta 1}(\mathbf{r}) = 0$, where $\mathbf{E}^2_{c\eta 1}(\mathbf{r})$ is given by Eq. (97), we find the identity

$$\begin{aligned}\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{R}})R^0_\eta = &\,\mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}}\sum_{b=\pm}\frac{1}{bK_z + k_{1z}} \\ &\times \sum_{mn}\Big\{[\mathbf{m}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}}) \otimes \mathbf{m}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T_n^1 x^1_{b\eta n} \\ &+ [\mathbf{n}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}}) \otimes \mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T_n^2 x^2_{b\eta n}\Big\}. \end{aligned} \tag{140}$$

Setting $z = 0$ in Eq. (139), using Eq. (140) and the fact that the field $\mathbf{E}^1_{c\eta 1}(\mathbf{r})$ as given by Eq. (96) vanishes, we obtain

$$\begin{aligned}\mathbf{E}_{c\eta}(\mathbf{r}_\perp, z = 0) &= \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}})\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp} + \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{R}})R^0_\eta\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp} \\ &= \mathbf{E}_{0\eta}(\mathbf{r}_\perp, z = 0) + \mathbf{E}_{\mathrm{R}\eta}(\mathbf{r}_\perp, z = 0). \end{aligned} \tag{141}$$

2. Case $z = H$. Using the expression of the coherent transmitted field $\mathbf{E}_{\mathrm{T}\eta}(\mathbf{r})$ as given by Eqs. (73) and (74), and the result $\mathbf{E}^1_{c\eta 1}(\mathbf{r}) = 0$, we find the identity

$$\begin{aligned}\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}})T^2_\eta\mathrm{e}^{\mathrm{j}k_{1z}H} = &-\mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}}\sum_{b=\pm}\frac{\mathrm{e}^{\mathrm{j}bK_zH}}{bK_z - k_{1z}} \\ &\times \sum_{mn}\Big\{[\mathbf{m}_{mn}(\widehat{\mathbf{s}}) \otimes \mathbf{m}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T_n^1 x^1_{b\eta n} \\ &+ [\mathbf{n}_{mn}(\widehat{\mathbf{s}}) \otimes \mathbf{n}_{-mn}(\widehat{\mathbf{s}}_b)]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_b)T_n^2 x^2_{b\eta n}\Big\}, \end{aligned} \tag{142}$$

so that, by setting $z = H$ in Eq. (139), using Eq. (142) and the result $\mathbf{E}^2_{c\eta 1}(\mathbf{r}) = 0$, we obtain

$$\mathbf{E}_{c\eta}(\mathbf{r}_\perp, z = H) = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}})T^2_\eta\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp}\mathrm{e}^{\mathrm{j}k_{1z}H} = \mathbf{E}_{\mathrm{T}\eta}(\mathbf{r}_\perp, z = H). \tag{143}$$

Thus, from Eqs. (141) and (143) it is readily seen that the coherent field $\mathbf{E}_{c\eta}(\mathbf{r})$ satisfies the continuity conditions for the electric fields

$$\mathbf{E}_{c\eta}(\mathbf{r}) = \mathbf{E}_{\mathrm{R}\eta}(\mathbf{r}) + \mathbf{E}_{0\eta}(\mathbf{r}) \text{ at } z = 0 \tag{144}$$

and

$$\mathbf{E}_{c\eta}(\mathbf{r}) = \mathbf{E}_{\mathrm{T}\eta}(\mathbf{r}) \text{ at } z = H. \tag{145}$$

From Eqs. (139), (141), and (143), it is straightforward to show that $\mathbf{E}_{c\eta}(\mathbf{r})$ can first be written as

$$\begin{aligned}\mathbf{E}_{c\eta}(\mathbf{r}) = &\,\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}})\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp}(A_\eta\mathrm{e}^{\mathrm{j}K_z z} + B_\eta\mathrm{e}^{-\mathrm{j}K_z z}) \\ &+ \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{R}})\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{r}_\perp}(C_\eta\mathrm{e}^{\mathrm{j}K_z z} + D_\eta\mathrm{e}^{-\mathrm{j}K_z z}), \end{aligned} \tag{146}$$

where

$$A_\eta = \frac{1}{1 - \mathrm{e}^{2\mathrm{j}K_zH}}\left[1 - T^2_\eta\mathrm{e}^{\mathrm{j}(k_{1z}+K_z)H}\right], \tag{147}$$

$$B_\eta = -\frac{1}{1 - \mathrm{e}^{2\mathrm{j}K_zH}}\left[\mathrm{e}^{2\mathrm{j}K_zH} - T^2_\eta\mathrm{e}^{\mathrm{j}(k_{1z}+K_z)H}\right], \tag{148}$$

$$C_\eta = \frac{1}{1 - \mathrm{e}^{2\mathrm{j}K_zH}}R^0_\eta, \tag{149}$$

$$D_\eta = -\frac{\mathrm{e}^{2\mathrm{j}K_zH}}{1 - \mathrm{e}^{2\mathrm{j}K_zH}}R^0_\eta, \tag{150}$$

and then as

$$\begin{aligned}\mathbf{E}_{c\eta}(\mathbf{r}) = &\,[\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{T}})T^1_\eta + \triangle T^1_\eta\widehat{\mathbf{z}}]\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{r}}\mathrm{e}^{\mathrm{j}K_z z} \\ &+ [\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{TR}})R^1_\eta + \triangle R^1_\eta\widehat{\mathbf{z}}]\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{r}}\mathrm{e}^{-\mathrm{j}K_z z}, \end{aligned} \tag{151}$$

where

$$\triangle T^1_\varphi = \triangle R^1_\varphi = 0 \tag{152}$$

and

$$\begin{aligned}\triangle T^1_\theta = &-\frac{1}{1 - \mathrm{e}^{2\mathrm{j}K_zH}}\left\{\left[1 + R^0_\theta - T^2_\theta\mathrm{e}^{\mathrm{j}(K_z+k_{1z})H}\right]\sin\theta_0\right. \\ &\left. - \left[1 - R^0_\theta - T^2_\theta\mathrm{e}^{\mathrm{j}(K_z+k_{1z})H}\right]\sin\theta_{\mathrm{T}}\right\}, \end{aligned} \tag{153}$$

$$\begin{aligned}\triangle R^1_\theta = &\,\frac{1}{1 - \mathrm{e}^{2\mathrm{j}K_zH}}\left\{\left[(1 + R^0_\theta)\mathrm{e}^{2\mathrm{j}K_zH} - T^2_\theta\mathrm{e}^{\mathrm{j}(K_z+k_{1z})H}\right]\sin\theta_0\right. \\ &\left. + \left[(1 - R^0_\theta)\mathrm{e}^{2\mathrm{j}K_zH} - T^2_\theta\mathrm{e}^{\mathrm{j}(K_z+k_{1z})H}\right]\sin\theta_{\mathrm{T}}\right\}. \end{aligned} \tag{154}$$

From the above analysis the following conclusions can be drawn.

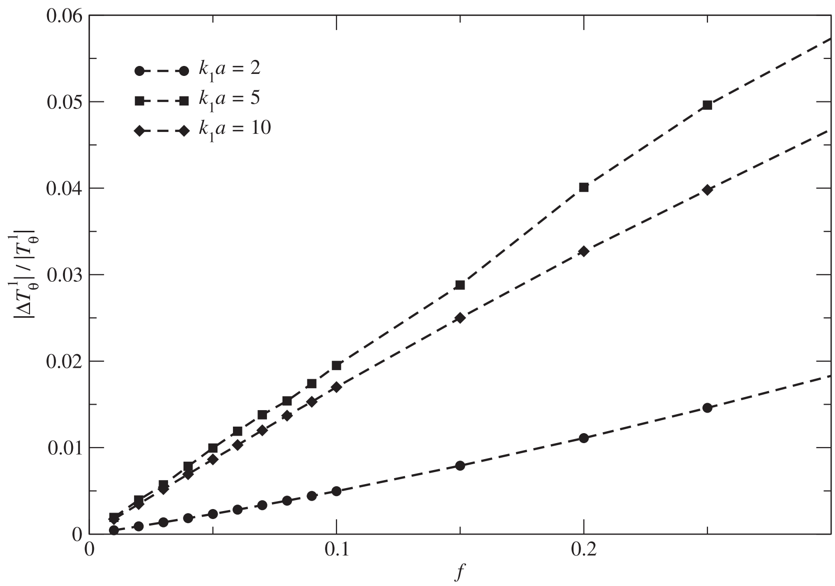

Fig. 7. The ratio $|\triangle T_\theta^1|/|T_\theta^1|$ as a function of the volume concentration f for three values of the size parameter k_1a. The layer thickness is $H = 50a$, the incidence angle is $\theta_0 = 30^\circ$, the wavenumber of the background medium is $k_1 = 10\,\mu\mathrm{m}^{-1}$, the relative refractive index of the particles is m = 1.33, and the maximum expansion order is $N_{\mathrm{rank}} = 15$. The ratio $|\triangle R_\theta^1|/|R_\theta^1|$ has a similar dependency on f.

1. The electric fields are continuous across the boundaries (not only their tangential components).
2. For normal incidence, as well as for a φ-polarized incidence, the representations (128) and (151) for the coherent field coincide, i.e., the coherent field is a superposition of plane electromagnetic waves with the wavenumber K.
3. For an oblique θ-polarized incidence, the coherent field satisfies the vector Helmholtz equation but is not divergence free, i.e., the coherent field is not a superposition of plane electromagnetic waves with a wavenumber K. However, as it can be infered from Fig. 7, $\triangle T_\theta^1$ and $\triangle R_\theta^1$ are small compared to T_θ^1 and R_θ^1, respectively; in the case $k_1a = 5$, we have $|\triangle T_\theta^1| < 2\times 10^{-3}|T_\theta^1|$ for $f = 0.01$ and $|\triangle T_\theta^1| < 0.01|T_\theta^1|$ for $f = 0.05$, while in the case $k_1a = 10$, we have $|\triangle T_\theta^1| < 1.5\times 10^{-3}|T_\theta^1|$ for $f \le 0.01$ and $|\triangle T_\theta^1| < 8\times 10^{-3}|T_\theta^1|$ for $f = 0.05$. Thus, for small volume concentrations, the coherent field can be assumed to be approximately divergence free.

3.4. Normal incidence

Let us particularize the above findings to the case of normal incidence.

Using the special values of the functions M_n and N_n at $x = 1$ as given by Eq. (121), as well as their special values at $x = -1$ given by

$$M_n(-1) = N_n(-1) = -(-1)^n \omega_n, \tag{155}$$

we find that for a θ-polarized incidence, Eqs. (33) and (34) of the generalized Ewald–Oseen extinction theorem become

$$1 = -\mathrm{j}\frac{\pi n_0}{k_1^2}\sum_n (2n+1)\Big[\frac{1}{K-k_1}(T_n^1 x_{+\theta n}^1 + T_n^2 x_{+\theta n}^2) + \frac{(-1)^n}{K+k_1}(T_n^1 x_{-\theta n}^1 - T_n^2 x_{-\theta n}^2)\Big], \tag{156}$$

$$0 = \sum_n (2n+1)\Big[\frac{(-1)^n \mathrm{e}^{\mathrm{j}KH}}{K+k_1}(T_n^1 x_{+\theta n}^1 - T_n^2 x_{+\theta n}^2) + \frac{\mathrm{e}^{-\mathrm{j}KH}}{K-k_1}(T_n^1 x_{-\theta n}^1 + T_n^2 x_{-\theta n}^2)\Big], \tag{157}$$

while for a φ-polarized incidence, Eqs. (35) and (36) become

$$1 = -\mathrm{j}\frac{\pi n_0}{k_1^2}\sum_n (2n+1)\Big[\frac{1}{K-k_1}(T_n^1 x_{+\varphi n}^1 + T_n^2 x_{+\varphi n}^2) - \frac{(-1)^n}{K+k_1}(T_n^1 x_{-\varphi n}^1 - T_n^2 x_{-\varphi n}^2)\Big], \tag{158}$$

$$0 = \sum_n (2n+1)\Big[\frac{(-1)^n \mathrm{e}^{\mathrm{j}KH}}{K+k_1}(T_n^1 x_{+\varphi n}^1 - T_n^2 x_{+\varphi n}^2) - \frac{\mathrm{e}^{-\mathrm{j}KH}}{K-k_1}(T_n^1 x_{-\varphi n}^1 + T_n^2 x_{-\varphi n}^2)\Big]. \tag{159}$$

Substituting Eq. (155) in Eqs. (63) and (64) gives

$$R_\theta^0 = -\mathrm{j}\frac{\pi n_0}{k_1^2(K+k_1)}\Big(1 - \mathrm{e}^{2\mathrm{j}KH}\Big) \times \sum_n (2n+1)(-1)^n (T_n^1 x_{+\theta n}^1 - T_n^2 x_{+\theta n}^2), \tag{160}$$

$$R_\varphi^0 = \mathrm{j}\frac{\pi n_0}{k_1^2(K+k_1)}\Big(1 - \mathrm{e}^{2\mathrm{j}KH}\Big) \times \sum_n (2n+1)(-1)^n (T_n^1 x_{+\varphi n}^1 - T_n^2 x_{+\varphi n}^2), \tag{161}$$

while substituting Eqs. (121) and (155) in Eqs. (77) and (78) gives

$$T_\theta^2 = -\mathrm{j}\frac{\pi n_0}{k_1^2}\sum_n (2n+1)\Big[\frac{1}{K-k_1}\mathrm{e}^{\mathrm{j}(K-k_1)H}(T_n^1 x_{+\theta n}^1 + T_n^2 x_{+\theta n}^2) + \frac{(-1)^n}{K+k_1}\mathrm{e}^{-\mathrm{j}(K+k_1)H}(T_n^1 x_{-\theta n}^1 - T_n^2 x_{-\theta n}^2)\Big], \tag{162}$$

$$T_\varphi^2 = -\mathrm{j}\frac{\pi n_0}{k_1^2}\sum_n (2n+1)\Big[\frac{1}{K-k_1}\mathrm{e}^{\mathrm{j}(K-k_1)H}(T_n^1 x_{+\varphi n}^1 + T_n^2 x_{+\varphi n}^2) - \frac{(-1)^n}{K+k_1}\mathrm{e}^{-\mathrm{j}(K_z+k_{1z})H}(T_n^1 x_{-\varphi n}^1 - T_n^2 x_{-\varphi n}^2)\Big]. \tag{163}$$

The reflection and transmission coefficients for the coherent field inside the layer are then obtained by inserting Eqs. (160)–(163) in Eqs. (129)–(132).

Note that in view of the result $|x^{1,2}_{-\eta n}| \ll |x^{1,2}_{+\eta n}|$, Eqs. (156)–(159) yield $x^{1,2}_{+\theta n} = x^{1,2}_{+\varphi n} = x^{1,2}_{+n}$. Hence, as in the effective field approximation, we have $R^0_\varphi = -R^0_\theta$ and $T^2_\varphi = T^2_\theta$, as well as $R^1_\varphi = -R^1_\theta$ and $T^1_\varphi = T^1_\theta$.

3.5. Semi-infinite discrete random medium

In the limit $H \to \infty$, the above formalism reduces to that of a semi-infinite discrete random medium. Actually, because $K''_z > 0$, we have $\mathrm{e}^{\mathrm{j}K_z H}\mathsf{e}_+ \to 0$ as $H \to \infty$. Employing this result in Eq. (15) and taking into account the representations of the elements of the matrix J^b_{2zH} given by Eqs. (133) and (134) of Ref. [1], we find that $\mathrm{e}^{-\mathrm{j}K_z H}\mathsf{e}_- \to 0$ as $H \to \infty$. Consequently, $\mathsf{e}_- \to 0$ as $H \to \infty$, and the equations of the Lorenz–Lorentz law (13) and the generalized Ewald–Oseen extinction theorem (14) become

$$\mathsf{e}_+ = n_0(\mathsf{J}^+_{1a} + \mathsf{J}^+_{2a})\mathsf{T}\mathsf{e}_+ \tag{164}$$

and

$$\mathsf{e}_0 + n_0\mathsf{J}^+_{2z0}\mathsf{T}\mathsf{e}_+ = 0, \tag{165}$$

respectively. Further simplifications are listed below.

1. The generalized Ewald–Oseen extinction theorem (165) gives
$$1 = -\mathrm{j}\frac{8\pi^2}{k_1 k_{1z}(K_z - k_{1z})}\sum_n \chi_n\sqrt{n(n+1)} \times [M_n(\widehat{\mathbf{s}}\cdot\widehat{\mathbf{s}}_{\mathrm{T}})T^1_n x^1_{+\theta n} - N_n(\widehat{\mathbf{s}}\cdot\widehat{\mathbf{s}}_{\mathrm{T}})T^2_n x^2_{+\theta n}] \tag{166}$$
for a θ-polarized incidence, and
$$1 = \mathrm{j}\frac{8\pi^2}{k_1 k_{1z}(K_z - k_{1z})}\sum_n \chi_n\sqrt{n(n+1)} \times [N_n(\widehat{\mathbf{s}}\cdot\widehat{\mathbf{s}}_{\mathrm{T}})T^1_n x^1_{+\varphi n} - M_n(\widehat{\mathbf{s}}\cdot\widehat{\mathbf{s}}_{\mathrm{T}})T^2_n x^2_{+\varphi n}] \tag{167}$$
for a φ-polarized incidence. For normal incidence, $x^{1,2}_{+\theta n} = x^{1,2}_{+\varphi n} = x^{1,2}_{+n}$, and the above equations reduces to (cf. Eqs. (156) and (158))
$$K = k_1 - \mathrm{j}\frac{\pi n_0}{k_1^2}\sum_n (2n+1)(T^1_n x^1_{+n} + T^2_n x^2_{+n}). \tag{168}$$
2. For the coherent field reflected by the layer (cf. Eq. (57)),
$$\mathbf{E}_{\mathrm{R}\eta}(\mathbf{r}) = \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}(K_z + k_{1z})}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{R}}\cdot\mathbf{r}} \times \sum_{mn}\{[\mathbf{m}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}}) \otimes \mathbf{m}_{-mn}(\widehat{\mathbf{s}}_{\mathrm{T}})]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{T}})T^1_n x^1_{+\eta n} + [\mathbf{n}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}}) \otimes \mathbf{n}_{-mn}(\widehat{\mathbf{s}}_{\mathrm{T}})]\cdot\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{T}})T^2_n x^2_{+\eta n}\}, \tag{169}$$
the reflection coefficients are given by (cf. Eqs. (63) and (64))
$$R^0_\theta = \mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}(K_z + k_{1z})}\sum_n \chi_n\sqrt{n(n+1)} \times [M_n(\widehat{\mathbf{s}}_{\mathrm{R}}\cdot\widehat{\mathbf{s}}_{\mathrm{T}})T^1_n x^1_{+\theta n} - N_n(\widehat{\mathbf{s}}_{\mathrm{R}}\cdot\widehat{\mathbf{s}}_{\mathrm{T}})T^2_n x^2_{+\theta n}] \tag{170}$$
and
$$R^0_\varphi = -\mathrm{j}\frac{8\pi^2 n_0}{k_1 k_{1z}(K_z + k_{1z})}\sum_n \chi_n\sqrt{n(n+1)} \times \left[N_n(\widehat{\mathbf{s}}_{\mathrm{R}}\cdot\widehat{\mathbf{s}}_{\mathrm{T}})T^1_n x^1_{+\varphi n} - M_n(\widehat{\mathbf{s}}_{\mathrm{R}}\cdot\widehat{\mathbf{s}}_{\mathrm{T}})T^2_n x^2_{+\varphi n}\right]. \tag{171}$$
3. Setting $x^{1,2}_{-\eta n} = 0$ in Eqs. (77) and (78), we find in the limit $H \to \infty$ that $T^2_\theta = T^2_\varphi = 0$, that is, the coherent field transmitted by the layer vanishes. Consequently,
$$T^1_\theta = (1 - R^0_\theta)\frac{\cos\theta_0}{\cos\theta_{\mathrm{T}}}, \quad T^1_\varphi = 1 + R^0_\varphi, \tag{172}$$
and $R^1_\theta = R^1_\varphi = 0$; and furthermore,
$$\mathbf{E}_{\mathrm{c}\eta}(\mathbf{r}) = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{T}})T^1_\eta \mathrm{e}^{\mathrm{j}K\widehat{\mathbf{s}}_{\mathrm{T}}\cdot\mathbf{r}}. \tag{173}$$
Note that the coherent field computed under the sparse-medium approximation for the integration domain is (cf. Eq. (151))
$$\mathbf{E}_{\mathrm{c}\eta}(\mathbf{r}) = [\widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{T}})T^1_\eta + \triangle T^1_\eta \widehat{\mathbf{z}}]\mathrm{e}^{\mathrm{j}K\widehat{\mathbf{s}}_{\mathrm{T}}\cdot\mathbf{r}}, \tag{174}$$
with $\triangle T^1_\varphi = 0$ and
$$\triangle T^1_\theta = (1 - R^0_\theta)\sin\theta_{\mathrm{T}} - (1 + R^0_\theta)\sin\theta_0. \tag{175}$$

4. Coherent field in a sparse medium

In principle, expressions for the effective wavenumber K and the coherent field $\mathbf{E}_{\mathrm{c}}(\mathbf{r})$ can be obtained by particularizing the results for a semi-infinite discrete random medium in the case of low volume concentration f. The procedure is as follows.

1. In Eq. (168), we approximate $x^{1,2}_{+n} \approx 1$ for $f \ll 1$, and find that the effective wavenumber K can be computed as
$$K = k_1 - \mathrm{j}\frac{\pi n_0}{k_1^2}\sum_n (2n+1)(T^1_n + T^2_n). \tag{176}$$
In view of Eqs. (18) and (19) and Eqs. (24) and (25), these approximations imply $e^1_{+\eta mn} \approx e^1_{0\eta mn}$ and $e^2_{+\eta mn} \approx e^2_{0\eta mn}$; this means that for sparse media, the conditional configuration-averaged exciting field can be considered to be approximately equal to the incident field [6].
2. In Eq. (172), we approximate $\theta_{\mathrm{T}} \approx \theta_0$ and $R^0_\eta \approx 0$; then $T^1_\eta \approx 1$, and so, from Eq. (173), $\mathbf{E}_{\mathrm{c}\eta}(\mathbf{r}) = \widehat{\boldsymbol{\eta}}(\widehat{\mathbf{s}}_{\mathrm{T}})\exp(\mathrm{j}K\widehat{\mathbf{s}}_{\mathrm{T}}\cdot\mathbf{r})$. This result implies that in the case of an incident wave of arbitrary polarization, we have
$$\mathbf{E}_{\mathrm{c}}(\mathbf{r}) = \mathrm{e}^{\mathrm{j}K\widehat{\mathbf{s}}_{\mathrm{T}}\cdot\mathbf{r}}\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}_{\mathrm{T}}), \tag{177}$$
where $\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}_{\mathrm{T}}) = \mathcal{E}_{0\theta}\widehat{\boldsymbol{\theta}}(\widehat{\mathbf{s}}_{\mathrm{T}}) + \mathcal{E}_{0\varphi}\widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{s}}_{\mathrm{T}})$. Under the same approximations, Eq. (175) yields $\triangle T^1_\theta = 0$, and the coherent field computed under the sparse-medium approximation for the integration domain is as in Eq. (177). Thus, the inconsistency between the two states of polarization of this field disappears.

It is interesting to mention that in the case of a layer of finite geometrical thickness H and normal incidence, the assumption $|x^{1,2}_{-\eta n}| \ll |x^{1,2}_{+\eta n}|$ implies

$$T^2_\theta = T^2_\varphi = -\mathrm{j}\frac{\pi n_0}{k_1^2}\frac{1}{K - k_1}\mathrm{e}^{\mathrm{j}(K-k_1)H}\sum_n (2n+1)(T^1_n x^1_{+n} + T^2_n x^2_{+n}), \tag{178}$$

so that from Eq. (176) and the approximation $x^{1,2}_{+n} \approx 1$, we find

$$T^2_\theta = T^2_\varphi = \mathrm{e}^{\mathrm{j}(K-k_1)H}. \tag{179}$$

Hence, by using the result $K - K^\star = \mathrm{j}n_0 C_{\mathrm{ext}}$, where C_{ext} is the extinction cross section of a spherical particle, we are led to the celebrated Bouguer exponential attenuation law [13] for the transmissivity T of the layer [11]:

$$T = (|T^2_\theta|)^2 = (|T^2_\varphi|)^2 = \mathrm{e}^{\mathrm{j}(K-K^\star)H} = \mathrm{e}^{-n_0 C_{\mathrm{ext}} H}. \tag{180}$$

In the following we intend to derive a self-contained theory for a sparse medium by starting from the initial formulae (11) and (44). Since the volume concentration is low, we (i) assume that the positions of the particles are uncorrelated ($g(R_{ij}) = 1$), and (ii) use the far-field approximation for the fields. Note that these assumptions are standard in deriving the vector radiative transfer equation. With these simplifications, the solution method is as follows.

1. We consider the integral equation (11) with $g(R_{ij}) = 1$ and the sparse-medium approximation for the integration domain, that is,

$$\langle \mathsf{e}_i \rangle_i = \mathsf{e}_{0i} + n_0 \int_D \mathsf{Q}(k_1 \mathbf{R}_{ij}) \langle \mathsf{e}_j \rangle_j \, \mathrm{d}^3\mathbf{R}_j, \tag{181}$$

where the incident field coefficients are given by

$$\mathsf{e}_{0i} = \mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{s}}\cdot\mathbf{R}_i} \mathsf{e}_0, \tag{182}$$

$$\mathsf{e}_0 = 4\pi \mathbf{x}^\star(\widehat{\mathbf{s}}) \cdot \boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}) = 4\pi [\mathsf{x}_\theta^\star(\widehat{\mathbf{s}})\mathcal{E}_{0\theta} + \mathsf{x}_\varphi^\star(\widehat{\mathbf{s}})\mathcal{E}_{0\varphi}] \tag{183}$$

with $\boldsymbol{\mathcal{E}}_0(\widehat{\mathbf{s}}) = \mathcal{E}_{0\theta}\widehat{\boldsymbol{\theta}}(\widehat{\mathbf{s}}) + \mathcal{E}_{0\varphi}\widehat{\boldsymbol{\varphi}}(\widehat{\mathbf{s}})$. Employing the far-field representation for the matrix $\mathsf{Q}(k_1\mathbf{R}_{ij})$ (cf. Eq. (45) of Ref. [1])

$$\mathsf{Q}(k_1\mathbf{R}_{ij}) = \frac{1}{k_1} g_0(R_{ij}) \mathsf{Q}_\infty(\widehat{\mathbf{R}}_{ij}), \tag{184}$$

where $g_0(R_{ij}) = \exp(\mathrm{j}k_1 R_{ij})/R_{ij}$ and (cf. Eq. (46) of Ref. [1])

$$\begin{aligned} \mathsf{Q}_\infty(\widehat{\mathbf{R}}_{ij}) &= -4\pi \mathrm{j} \mathbf{x}^\star(\widehat{\mathbf{R}}_{ij}) \cdot \mathbf{x}^T(\widehat{\mathbf{R}}_{ij}) \mathsf{T} \\ &= -4\pi \mathrm{j} [\mathsf{x}_\theta^\star(\widehat{\mathbf{R}}_{ij}) \mathsf{x}_\theta^T(\widehat{\mathbf{R}}_{ij}) + \mathsf{x}_\varphi^\star(\widehat{\mathbf{R}}_{ij}) \mathsf{x}_\varphi^T(\widehat{\mathbf{R}}_{ij})] \mathsf{T}, \end{aligned} \tag{185}$$

we compute the conditional configuration-averaged exciting field coefficients from the integral equation

$$\langle \mathsf{e}_i \rangle_i = \mathsf{e}_{0i} + \frac{n_0}{k_1} \int_D g_0(R_{ij}) \mathsf{Q}_\infty(\widehat{\mathbf{R}}_{ij}) \langle \mathsf{e}_j \rangle_j \, \mathrm{d}^3\mathbf{R}_j. \tag{186}$$

2. In the integral representation (cf. Eq. (44) with the sparse-medium approximation for the integration domain)

$$\mathbf{E}_c(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + n_0 \int_D \mathbf{X}_3^T(k_1\mathbf{r}_i) \mathsf{T} \langle \mathsf{e}_i \rangle_i \, \mathrm{d}^3\mathbf{R}_i, \tag{187}$$

we use the far-field approximation for the radiating vector spherical wave functions (cf. Eq. (11) of Ref. [1])

$$\mathbf{X}_3(k_1\mathbf{r}_i) = -\frac{\mathrm{j}}{k_1} g_0(r_i) \mathbf{x}(\widehat{\mathbf{r}}_i),$$

and compute the coherent field according to

$$\mathbf{E}_c(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) - n_0 \frac{\mathrm{j}}{k_1} \int_D g_0(r_i) \mathbf{x}^T(\widehat{\mathbf{r}}_i) \mathsf{T} \langle \mathsf{e}_i \rangle_i \, \mathrm{d}^3\mathbf{R}_i. \tag{188}$$

The procedure of configuration averaging requires the computation of integrals over particle positions. To integrate over all positions of particle i we use a local coordinate system with the origin at the observation point; to integrate over all positions of particle j we use a local coordinate system with the origin at particle i; and so on. In other words, we make the changes of variables

$$\mathbf{R}_i = \mathbf{r} + \mathbf{p}, \quad \mathbf{R}_j = \mathbf{R}_i + \mathbf{R}_{ji}, \ldots \tag{189}$$

The integration domain is D without any exclusion sphere. As shown in Fig. 8, for direction $\widehat{\mathbf{p}}$, p ranges from zero at the observation point to the corresponding value at the point C (where the straight line with the direction vector $\widehat{\mathbf{p}}$ crosses the lower plane boundary); for direction $\widehat{\mathbf{R}}_{ji}$, R_{ji} ranges from zero at the origin of particle i to the corresponding value at the point C_i (where the straight line with the direction vector $\widehat{\mathbf{R}}_{ji}$ crosses the lower plane boundary), etc.

4.1. Conditional configuration-averaged exciting field coefficients

The iterated solution of Eq. (186) is

$$\begin{aligned} \langle \mathsf{e}_i \rangle_i = \mathsf{e}_{0i} &+ \frac{n_0}{k_1} \int_D g_0(R_{ij}) \mathsf{Q}_\infty(\widehat{\mathbf{R}}_{ij}) \mathsf{e}_{0j} \, \mathrm{d}^3\mathbf{R}_j \\ &+ \frac{n_0^2}{k_1^2} \int_D g_0(R_{ij}) g_0(R_{jk}) \mathsf{Q}_\infty(\widehat{\mathbf{R}}_{ij}) \mathsf{Q}_\infty(\widehat{\mathbf{R}}_{jk}) \mathsf{e}_{0k} \, \mathrm{d}^3\mathbf{R}_k \mathrm{d}^3\mathbf{R}_j + \cdots. \end{aligned} \tag{190}$$

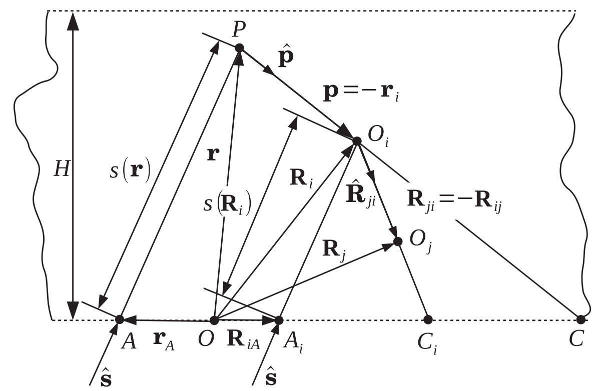

Fig. 8. Scattering geometry for computing the coherent field in the case of a sparse discrete random medium.

Our objective is to compute each integral in Eq. (190) and then to sum up the resulting series.

Consider the first integral in Eq. (190) and denote it by $\mathsf{I}_1(\mathbf{R}_i)$. Making the change of variable (cf. Eq. (189)) $\mathbf{R}_j = \mathbf{R}_i + \mathbf{R}_{ji}$ and using the asymptotic spherical-wave expansion of the plane wave $\exp(\mathrm{j}k_1\widehat{\mathbf{s}} \cdot \mathbf{R}_{ji})$ [14],

$$\mathrm{e}^{\mathrm{j}k\widehat{\mathbf{s}}\cdot\mathbf{R}_{ji}} = \mathrm{j}\frac{2\pi}{k_1 R_{ji}} \left[\delta(\widehat{\mathbf{s}} + \widehat{\mathbf{R}}_{ji}) \mathrm{e}^{-\mathrm{j}k_1 R_{ji}} - \delta(\widehat{\mathbf{s}} - \widehat{\mathbf{R}}_{ji}) \mathrm{e}^{\mathrm{j}k_1 R_{ji}} \right] \tag{191}$$

as $k_1 R_{ji} \to \infty$, where $\delta(\widehat{\mathbf{s}})$ is the solid-angle delta function, we obtain

$$\begin{aligned} \mathsf{I}_1(\mathbf{R}_i) &= \left(\mathrm{j}\frac{2\pi}{k_1^2} n_0 \right) \mathrm{e}^{\mathrm{j}k_1\widehat{\mathbf{s}}\cdot\mathbf{R}_i} \bigg[\mathsf{Q}_\infty(\widehat{\mathbf{s}}) \mathsf{e}_0 \int_0^s \mathrm{d}R_{ji} \\ &\quad - \mathsf{Q}_\infty(-\widehat{\mathbf{s}}) \mathsf{e}_0 \int_0^{\bar{s}} \mathrm{e}^{2\mathrm{j}k_1 R_{ji}} \mathrm{d}R_{ji} \bigg] \\ &= \left(\mathrm{j}\frac{2\pi}{k_1^2} n_0 \right) \mathrm{e}^{\mathrm{j}k_1\widehat{\mathbf{s}}\cdot\mathbf{R}_i} \left[s\mathsf{Q}_\infty(\widehat{\mathbf{s}}) - \frac{\mathrm{e}^{2\mathrm{j}k_1\bar{s}} - 1}{2\mathrm{j}k_1} \mathsf{Q}_\infty(-\widehat{\mathbf{s}}) \right] \mathsf{e}_0. \end{aligned} \tag{192}$$

Here, $s = s(\mathbf{R}_i, -\widehat{\mathbf{s}}) = \widehat{\mathbf{s}} \cdot (\mathbf{R}_i - \mathbf{R}_{A_i})$ and $\bar{s} = s(\mathbf{R}_i, \widehat{\mathbf{s}}) = \widehat{\mathbf{s}} \cdot (\mathbf{R}_{A_i'} - \mathbf{R}_i)$, where $\mathbf{R}_{A_i}$ and $\mathbf{R}_{A_i'}$ are the points where the straight lines with the direction vectors $-\widehat{\mathbf{s}}$ and $\widehat{\mathbf{s}}$ going through O_i cross the lower and the upper boundary of the layer, respectively. By assuming that the particles are separated by distances much larger than the wavelength, i.e., $k_1 s \gg 1$ (except for points in the immediate vicinity of the boundary), and taking account of

$$\left| \frac{\mathrm{e}^{2\mathrm{j}k_1\bar{s}} - 1}{2\mathrm{j}} \right| = \frac{1}{2} \left| \mathrm{e}^{2\mathrm{j}k_1\bar{s}} - 1 \right| \le \frac{1}{2} \left(\left| \mathrm{e}^{2\mathrm{j}k_1\bar{s}} \right| + 1 \right) = 1, \tag{193}$$

we find

$$s \gg \frac{1}{k_1} \ge \left| \frac{\mathrm{e}^{2\mathrm{j}k_1\bar{s}} - 1}{2\mathrm{j}k_1} \right|. \tag{194}$$

Thus, the second term on the right-hand side of Eq. (192) is much smaller than the first term, and the result is

$$\mathsf{I}_1(\mathbf{R}_i) = \left(\mathrm{j}\frac{2\pi}{k_1} n_0 s \right) \mathrm{e}^{\mathrm{j}k_1\widehat{\mathbf{s}}\cdot\mathbf{R}_i} \left[\frac{1}{k_1} \mathsf{Q}_\infty(\widehat{\mathbf{s}}) \right] \mathsf{e}_0. \tag{195}$$

For the second integral, denoted by $\mathsf{I}_2(\mathbf{R}_i)$, we change variables according to $\mathbf{R}_j = \mathbf{R}_i + \mathbf{R}_{ji}$ and $\mathbf{R}_k = \mathbf{R}_j + \mathbf{R}_{kj}$, employ the asymptotic spherical-waves representations for the plane waves $\exp(\mathrm{j}k_1\widehat{\mathbf{s}} \cdot \mathbf{R}_{ji})$ and $\exp(\mathrm{j}k_1\widehat{\mathbf{s}} \cdot \mathbf{R}_{kj})$, and hence derive

$$
\begin{aligned}
\mathrm{I}_2(\mathbf{R}_i) &= \frac{n_0^2}{k_1^2}\mathrm{e}^{\mathrm{j}k_1\hat{\mathbf{s}}\cdot\mathbf{R}_i}\int_D g_0(R_{ji})\mathrm{e}^{\mathrm{j}k_1\hat{\mathbf{s}}\cdot\mathbf{R}_{ji}}\mathrm{Q}_\infty(-\hat{\mathbf{R}}_{ji}) \\
&\times\left[\int_D g_0(R_{kj})\mathrm{e}^{\mathrm{j}k_1\hat{\mathbf{s}}\cdot\mathbf{R}_{kj}}\mathrm{Q}_\infty(-\hat{\mathbf{R}}_{kj})\mathrm{e}_0\,\mathrm{d}^3\mathbf{R}_{kj}\right]\mathrm{d}^3\mathbf{R}_{ji} \\
&= \left(\mathrm{j}\frac{2\pi}{k_1}n_0\right)^2\mathrm{e}^{\mathrm{j}k_1\hat{\mathbf{s}}\cdot\mathbf{R}_i}\left[\frac{1}{k_1}\mathrm{Q}_\infty(\hat{\mathbf{s}})\right]^2\mathrm{e}_0\int_0^s\left[\int_{R_{ji}}^s\mathrm{d}R_{kj}\right]\mathrm{d}R_{ji} \\
&= \frac{1}{2}\left(\mathrm{j}\frac{2\pi}{k_1}n_0 s\right)^2\mathrm{e}^{\mathrm{j}k_1\hat{\mathbf{s}}\cdot\mathbf{R}_i}\left[\frac{1}{k_1}\mathrm{Q}_\infty(\hat{\mathbf{s}})\right]^2\mathrm{e}_0. \qquad (196)
\end{aligned}
$$

Next, we use Eqs. (183) and (185), the expression of the elements of the amplitude matrix given by (cf. Eq. (13) of Ref. [1])

$$S_{\eta\mu}(\hat{\mathbf{s}},\hat{\mathbf{s}}) = -\frac{4\pi\mathrm{j}}{k_1}\mathrm{x}_\eta^T(\hat{\mathbf{s}})\mathrm{T}\mathrm{x}_\mu^\star(\hat{\mathbf{s}}),\quad \eta,\mu=\theta,\varphi, \qquad (197)$$

and the well-known identity for spherically symmetric particles

$$S_{\eta\mu}(\hat{\mathbf{s}},\hat{\mathbf{s}}) \equiv 0,\quad \eta\neq\mu, \qquad (198)$$

to derive

$$
\begin{aligned}
\mathrm{Q}_\infty(\hat{\mathbf{s}})\mathrm{e}_0 &= 4\pi k_1[\mathrm{x}_\theta^\star(\hat{\mathbf{s}})S_{\theta\theta}(\hat{\mathbf{s}},\hat{\mathbf{s}})\mathcal{E}_{0\theta}+\mathrm{x}_\varphi^\star(\hat{\mathbf{s}})S_{\varphi\varphi}(\hat{\mathbf{s}},\hat{\mathbf{s}})\mathcal{E}_{0\varphi}] \\
&= k_1 S(0)\mathrm{e}_0, \qquad (199)
\end{aligned}
$$

where

$$S(0) = S_{\eta\eta}(\hat{\mathbf{s}},\hat{\mathbf{s}}) = -\frac{\mathrm{j}}{2k_1}\sum_n(2n+1)(T_n^1+T_n^2),\quad \eta=\theta,\varphi. \qquad (200)$$

From Eq. (199), it is readily seen that in general,

$$\left[\frac{1}{k_1}\mathrm{Q}_\infty(\hat{\mathbf{s}})\right]^n\mathrm{e}_0 = S^n(0)\mathrm{e}_0,\quad n=1,2,\ldots, \qquad (201)$$

Proceeding similarly for all terms in the series (190), and accounting for Eq. (201), we find that the sum of the series (190) is

$$\langle\mathrm{e}_i\rangle_i = \exp\left\{\mathrm{j}\left[k_1\hat{\mathbf{s}}\cdot\mathbf{R}_i+\frac{2\pi}{k_1}n_0 S(0)s(\mathbf{R}_i,-\hat{\mathbf{s}})\right]\right\}\mathrm{e}_0. \qquad (202)$$

4.2. Coherent field

To compute the coherent field, we consider the integral representation (188). Using the series expansion (cf. Eq. (202))

$$\langle\mathrm{e}_i\rangle_i = \mathrm{e}^{\mathrm{j}k_1\hat{\mathbf{s}}\cdot\mathbf{R}_i}\left[1+\mathrm{j}\frac{2\pi}{k_1}n_0 S(0)s(\mathbf{R}_i,-\hat{\mathbf{s}})+\cdots\right]\mathrm{e}_0. \qquad (203)$$

the result

$$
\begin{aligned}
&\int_D g_0(r_i)\mathrm{e}^{\mathrm{j}k_1\hat{\mathbf{s}}\cdot\mathbf{R}_i}s^n(\mathbf{R}_i,-\hat{\mathbf{s}})\mathbf{x}^T(\hat{\mathbf{r}}_i)\mathrm{T}\mathrm{e}_0\,\mathrm{d}^3\mathbf{R}_i \\
&= \mathrm{j}\frac{2\pi}{k_1}\frac{s^{n+1}(\mathbf{r},-\hat{\mathbf{s}})}{n+1}\mathrm{e}^{\mathrm{j}k_1\hat{\mathbf{s}}\cdot\mathbf{r}}\mathbf{x}^T(\hat{\mathbf{s}})\mathrm{T}\mathrm{e}_0,\quad n=0,1,\ldots, \qquad (204)
\end{aligned}
$$

and the identity (cf. Eqs. (183), (197), and (198))

$$
\begin{aligned}
&-\frac{\mathrm{j}}{k_1}\mathbf{x}^T(\hat{\mathbf{s}})\mathrm{T}\mathrm{e}_0 = -\frac{4\mathrm{j}\pi}{k_1}\mathbf{x}^T(\hat{\mathbf{s}})\mathrm{T}\mathbf{x}^\star(\hat{\mathbf{s}})\cdot\mathcal{E}_0(\hat{\mathbf{s}}) \\
&= -\frac{4\mathrm{j}\pi}{k_1}[\mathrm{x}_\theta^T(\hat{\mathbf{s}})\hat{\boldsymbol{\theta}}(\hat{\mathbf{s}})+\mathrm{x}_\varphi^T(\hat{\mathbf{s}})\hat{\boldsymbol{\varphi}}(\hat{\mathbf{s}})]\mathrm{T}[\mathrm{x}_\theta^\star(\hat{\mathbf{s}})\mathcal{E}_{0\theta}+\mathrm{x}_\varphi^\star(\hat{\mathbf{s}})\mathcal{E}_{0\varphi}] \\
&= S(0)\mathcal{E}_0(\hat{\mathbf{s}}), \qquad (205)
\end{aligned}
$$

we end up with

$$\mathbf{E}_\mathrm{c}(\mathbf{r}) = \exp\left[\mathrm{j}\frac{2\pi}{k_1}n_0 S(0)s(\mathbf{r},-\hat{\mathbf{s}})\right]\mathbf{E}_0(\mathbf{r}). \qquad (206)$$

Here, $s(\mathbf{r},-\hat{\mathbf{s}})$ is defined through the relation $s(\mathbf{r},-\hat{\mathbf{s}})=\hat{\mathbf{s}}\cdot(\mathbf{r}-\mathbf{r}_A)$, where $\mathbf{r}_A$ is the point where the straight line parallel to the incidence direction and going through the observation point crosses the boundary of the medium (see Fig. 8).

Using the incident field representation

$$\mathbf{E}_0(\mathbf{r}) = \mathrm{e}^{\mathrm{j}k_1 s(\mathbf{r},-\hat{\mathbf{s}})}\mathbf{E}_0(\mathbf{r}_A), \qquad (207)$$

with $\mathbf{E}_0(\mathbf{r}_A)=\exp(\mathrm{j}k_1\hat{\mathbf{s}}\cdot\mathbf{r}_A)\mathcal{E}_0(\hat{\mathbf{s}})$, we express Eq. (206) as

$$\mathbf{E}_\mathrm{c}(\mathbf{r}) = \exp\left\{\mathrm{j}\left[k_1+\frac{2\pi}{k_1}n_0 S(0)\right]s(\mathbf{r},-\hat{\mathbf{s}})\right\}\mathbf{E}_0(\mathbf{r}_A). \qquad (208)$$

Let us define the wavenumber K by the relation (compare with Eq. (176)) [16]

$$K = k_1+\frac{2\pi}{k_1}n_0 S(0) = k_1-\mathrm{j}\frac{\pi n_0}{k_1^2}\sum_n(2n+1)(T_n^1+T_n^2). \qquad (209)$$

Then, from Eq. (208) we deduce that the coherent field is analytically equivalent to a plane electromagnetic wave with the wavenumber K, propagation direction $\hat{\mathbf{s}}$, and amplitude $\exp(-\mathrm{j}K\hat{\mathbf{s}}\cdot\mathbf{r}_A)\mathbf{E}_0(\mathbf{r}_A)$. Therefore, K can be interpreted as the effective wavenumber in the particulate medium. Setting $\mathbf{r}=\mathbf{r}_A$ in Eq. (208) and taking into account that $s(\mathbf{r}_A,-\hat{\mathbf{s}})=0$, we find the boundary condition

$$\mathbf{E}_\mathrm{c}(\mathbf{r}_A) = \mathbf{E}_0(\mathbf{r}_A), \qquad (210)$$

and so, the following equivalent representation for the coherent field

$$\mathbf{E}_\mathrm{c}(\mathbf{r}) = \mathrm{e}^{\mathrm{j}Ks(\mathbf{r},-\hat{\mathbf{s}})}\mathbf{E}_\mathrm{c}(\mathbf{r}_A). \qquad (211)$$

Thus, in contrast to a layer with densely packed spherical particles, the coherent field in a layer with a sparse concentration of particles is an upwelling wave. The upper boundary does not produce a downwelling wave, and the upwelling coherent field is uniquely determined by the boundary condition (210). It is for this reason that in the introductory part of the section, we computed the effective wavenumber and the coherent field by particularizing the results for a *semi-infinite* discrete random medium.

Let us define the wave vector $\mathbf{K}_0$ by

$$\mathbf{K}_0 = k_1\hat{\mathbf{s}}+\frac{2\pi}{k_1}n_0 S(0)\frac{\hat{\mathbf{z}}}{\cos\theta_0}, \qquad (212)$$

implying that (cf. Eq. (209))

$$\mathbf{K}_0 = k_1\hat{\mathbf{s}}+(K-k_1)\frac{\hat{\mathbf{z}}}{\cos\theta_0}. \qquad (213)$$

Using the relation $s(\mathbf{r},-\hat{\mathbf{s}})=\hat{\mathbf{z}}\cdot\mathbf{r}/\cos\theta_0$, we express Eqs. (202) and (208) in terms of the wave vector $\mathbf{K}_0$ as (compare with Eq. (12))

$$\langle\mathrm{e}_i\rangle_i = \mathrm{e}^{\mathrm{j}\mathbf{K}_0\cdot\mathbf{R}_i}\mathrm{e}_0 \qquad (214)$$

and

$$\mathbf{E}_\mathrm{c}(\mathbf{r}) = \mathrm{e}^{\mathrm{j}\mathbf{K}_0\cdot\mathbf{r}}\mathcal{E}_0(\hat{\mathbf{s}}), \qquad (215)$$

respectively. Thus, for a layer with a sparse concentration of particles, the effective wavenumber is computed from Eq. (209), the conditional configuration-averaged exciting field coefficients from Eq. (214), and the coherent field from Eq. (215).

Some comments are in order.

1. The effective wave vector $\mathbf{K}$, defined by $\mathbf{K}=K\hat{\mathbf{s}}_\mathrm{T}$, is related to $\mathbf{K}_0$ by the relation

   $$\mathbf{K}_0 = \mathbf{K}+\triangle K_z\hat{\mathbf{z}}, \qquad (216)$$

   where

   $$\triangle K_z = \frac{1-\cos(\theta_\mathrm{T}-\theta_0)}{\cos\theta_0}K. \qquad (217)$$

   Approximating $\theta_\mathrm{T}\approx\theta_0$, yields $\triangle K_z\approx 0$, and so, $\mathbf{K}\approx\mathbf{K}_0$. Therefore, $\mathbf{K}_0$ can be interpreted as the effective wave vector in a sparse medium. In view of the approximations $\mathbf{K}\approx\mathbf{K}_0$ and $\theta_\mathrm{T}\approx\theta_0$ implying $\hat{\mathbf{s}}_\mathrm{T}\approx\hat{\mathbf{s}}$, we deduce that the representations (177) and (215) for the coherent field are equivalent.

2. Setting $\mathbf{r} = \mathbf{R}_i$ in the representation of the coherent field as given by Eq. (215) yields $\mathbf{E}_c(\mathbf{R}_i) = \exp(\mathrm{j}\mathbf{K}_0 \cdot \mathbf{R}_i)\mathcal{E}_0(\widehat{\mathbf{s}})$, so that from Eq. (214) and the representation (cf. Eq. (183)) $\mathrm{e}_0 = 4\pi \mathbf{x}^\star(\widehat{\mathbf{s}}) \cdot \mathcal{E}_0(\widehat{\mathbf{s}})$, we find

$$\langle \mathrm{e}_i \rangle_i = 4\pi \mathbf{x}^\star(\widehat{\mathbf{s}}) \cdot \mathbf{E}_c(\mathbf{R}_i). \tag{218}$$

Inserting Eq. (218) in Eq. (188) and taking into account the representation for the far-field scattering dyadic (cf. Eq. (15) of Ref. [1])

$$\overline{\mathbf{A}}(\widehat{\mathbf{r}}, \widehat{\mathbf{s}}) = -\frac{4\pi \mathrm{j}}{k_1} \mathbf{x}^T(\widehat{\mathbf{r}}) \mathrm{T} \mathbf{x}^\star(\widehat{\mathbf{s}}),$$

we infer that for sparse media, the coherent field satisfies the integral equation

$$\mathbf{E}_c(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + n_0 \int g_0(r_i) \overline{\mathbf{A}}(\widehat{\mathbf{r}}_i, \widehat{\mathbf{s}}) \cdot \mathbf{E}_c(\mathbf{R}_i)\, \mathrm{d}^3\mathbf{R}_i. \tag{219}$$

This equation is the so-called Foldy integral equation for the coherent field.

In the following parts of this series, we will compute the second-order moment of the fields by approximating the conditional configuration-averaged exciting field coefficients of a layer with densely packed particles by those of a semi-infinite medium with densely packed particles. In this context, with K being calculated from the generalized Lorenz–Lorentz law for a dense semi-infinite discrete random medium at normal incidence, Tishkovets and Jockers [15] assumed that the sparse-medium approximation (214) is also valid in this case. The benefit of using Eq. (214) together with Eq. (213) is that the vector e_+, which determines $\langle \mathrm{e}_i \rangle_i$ according to the generalized Lorenz–Lorentz law and the generalized Ewald–Oseen extinction theorem for a dense semi-infinite discrete random medium, does not need to be computed.

4.3. The Twersky and Foldy approximations

In this section, we will show that the Twersky and Foldy approximations are implicit in our derivation.

Consider a configuration with $N-1$ particles Λ^i_{N-1} in which particle i is removed from the group. According to the Twersky approximation, the field exciting particle i at a point $\mathbf{r}_i$ near particle i is the total electric field that would exists at that point if the particle i were removed from the group, i.e.,

$$\mathbf{E}^{(i)}_{\mathrm{exc}\,i}(\mathbf{r}_i) = \mathbf{E}^{(i)}(\mathbf{r}_i | \Lambda^i_{N-1}). \tag{220}$$

Here, the superscript "*(i)*" indicates that the fields are written in the coordinate system of particle i. Taking the configuration average of Eq. (220) with the position of particle i held fixed, and using the configuration average rule

$$\langle f(\mathbf{r}, \Lambda_N) \rangle_i = \int f(\mathbf{r}, \Lambda_N) p(\Lambda^i_{N-1} | \mathbf{R}_i)\, \mathrm{d}\Lambda^i_{N-1}$$

yields

$$\langle \mathbf{E}^{(i)}_{\mathrm{exc}\,i}(\mathbf{r}_i) \rangle_i = \mathbf{E}^{(i)}_c(\mathbf{r}_i | \Lambda^i_{N-1}). \tag{221}$$

Thus, in the Twersky approximation, *the conditional configuration-averaged field exciting particle i at a point* $\mathbf{r}_i$ *near particle i is the coherent field that would exists at that point if particle i were removed from the group.* Let us prove this conjecture. The conditional configuration average of the field exciting particle i at the field point $\mathbf{r} = \mathbf{R}_i + \mathbf{r}_i$ is

$$\langle \mathbf{E}_{\mathrm{exc}\,i}(\mathbf{r}) \rangle_i = \langle \mathbf{E}^{(i)}_{\mathrm{exc}\,i}(\mathbf{r}_i) \rangle_i = \mathbf{X}^T_1(k_1 \mathbf{r}_i) \langle \mathrm{e}_i \rangle_i. \tag{222}$$

On the other hand, upon removing particle i from the group, the coherent field at the same field point $\mathbf{r}$ produced by the remaining $N-1$ particles, denoted by $\mathbf{E}_c(\mathbf{r} | \Lambda^i_{N-1})$, computes as ($n_0 \approx (N-1)/V$)

$$\mathbf{E}_c(\mathbf{r} | \Lambda^i_{N-1}) = \mathbf{E}_0(\mathbf{r}) + \sum_{j \neq i} \langle \mathbf{E}_{\mathrm{sct}\,j}(\mathbf{r}) \rangle = \mathbf{E}_0(\mathbf{r}) + n_0 \int_D \mathbf{X}^T_3(k_1 \mathbf{r}_j) \mathrm{T} \langle \mathrm{e}_j \rangle_j \mathrm{d}^3\mathbf{R}_j. \tag{223}$$

Choosing $\mathbf{r}_i$ in the neighborhood of particle i before the removal, we use the translation addition theorem $\mathbf{X}_3(k_1 \mathbf{r}_j) = \mathcal{T}_{31}(k_1 \mathbf{R}_{ij}) \mathbf{X}_1(k_1 \mathbf{r}_i)$ with $\mathbf{r}_j = \mathbf{r}_i + \mathbf{R}_{ij}$, the incident field representation $\mathbf{E}_0(\mathbf{r}) = \mathbf{E}^{(i)}_0(\mathbf{r}_i) = \exp(\mathrm{j}k_1 \widehat{\mathbf{s}} \cdot \mathbf{r}_i) \mathbf{E}_0(\mathbf{R}_i) = \mathbf{X}^T_1(k_1 \mathbf{r}_i) \mathrm{e}_{0i}$, and the integral equation for the conditional configuration-averaged exciting field coefficients (cf. Eq. (181)),

$$\langle \mathrm{e}_i \rangle_i = \mathrm{e}_{0i} + n_0 \int_D \mathcal{T}^T_{31}(k_1 \mathbf{R}_{ij}) \mathrm{T} \langle \mathrm{e}_j \rangle_j \mathrm{d}^3\mathbf{R}_j,$$

to obtain

$$\mathbf{E}_c(\mathbf{r} | \Lambda^i_{N-1}) = \mathbf{E}^{(i)}_c(\mathbf{r}_i | \Lambda^i_{N-1}) = \mathbf{X}^T_1(k_1 \mathbf{r}_i) \langle \mathrm{e}_i \rangle_i. \tag{224}$$

From Eqs. (222) and (224), the Twersky approximation (221) readily follows.

The Foldy approximation states that *the coherent field at a point* $\mathbf{r}_i$ *near particle i is the conditional configuration average of the field exciting particle i at that point*, or equivalently and in view of Eq. (221), that *the coherent field at a point* $\mathbf{r}_i$ *near particle i is the coherent field that would exist at that point if the particle i were removed from the group:*

$$\mathbf{E}^{(i)}_c(\mathbf{r}_i) = \langle \mathbf{E}^{(i)}_{\mathrm{exc}\,i}(\mathbf{r}_i) \rangle_i = \mathbf{E}^{(i)}_c(\mathbf{r}_i | \Lambda^i_{N-1}). \tag{225}$$

To prove this result we use Eq. (218), that is, $\langle \mathrm{e}_i \rangle_i = 4\pi \mathbf{x}^\star(\widehat{\mathbf{s}}) \cdot \mathbf{E}_c(\mathbf{R}_i)$, where (cf. Eqs. (213) and (215))

$$\mathbf{E}_c(\mathbf{R}_i) = \mathrm{e}^{\mathrm{j}(K-k_1)s(\mathbf{R}_i, -\widehat{\mathbf{s}})} \mathbf{E}_0(\mathbf{R}_i), \tag{226}$$

and Eq. (222) to conclude that $\langle \mathbf{E}^{(i)}_{\mathrm{exc}\,i}(\mathbf{r}_i) \rangle_i$ is analytically equivalent to the plane electromagnetic wave

$$\langle \mathbf{E}^{(i)}_{\mathrm{exc}\,i}(\mathbf{r}_i) \rangle_i = \mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{s}} \cdot \mathbf{r}_i} \mathbf{E}_c(\mathbf{R}_i) = \mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{s}} \cdot \mathbf{r}_i} \mathrm{e}^{\mathrm{j}(K-k_1)s(\mathbf{R}_i, -\widehat{\mathbf{s}})} \mathbf{E}_0(\mathbf{R}_i). \tag{227}$$

Apart from that, Eqs. (213) and (215) also give

$$\mathbf{E}_c(\mathbf{r}) = \mathbf{E}^{(i)}_c(\mathbf{r}_i) = \mathrm{e}^{\mathrm{j}k_1 \widehat{\mathbf{s}} \cdot \mathbf{r}_i} \mathrm{e}^{\mathrm{j}(K-k_1)s(\mathbf{r}, -\widehat{\mathbf{s}})} \mathbf{E}_0(\mathbf{R}_i). \tag{228}$$

If the point $\mathbf{r}_i$ is near particle i, and the boundary is in the far-field region of this particle, we approximate $s(\mathbf{R}_i, -\widehat{\mathbf{s}}) \approx s(\mathbf{r}, -\widehat{\mathbf{s}})$; hence from Eqs. (227) and (228), we obtain the Foldy approximation (225). It is interesting to note that the representation $\mathbf{E}^{(i)}_c(\mathbf{r}_i) = \exp(\mathrm{j}k_1 \widehat{\mathbf{s}} \cdot \mathbf{r}_i) \mathbf{E}_c(\mathbf{R}_i)$, which follows from Eq. (227), shows that *the coherent field near a particle can be analytically approximated by a plane electromagnetic wave with wavenumber* k_1 *and propagation direction* $\widehat{\mathbf{s}}$.

5. Discussion

The analysis performed in this paper is based on the assumption that the conditional configuration-averaged exciting field coefficients are of the form $\langle \mathrm{e}_i \rangle_i(\mathbf{R}_i) = \sum_{b=\pm} \exp(\mathrm{j}\mathbf{K}_b \cdot \mathbf{R}_i) \mathrm{e}_b$, and is restricted to the computation of the zeroth-order fields without a special treatment of the critical domains.

In this setting we have calculated the coherent fields reflected and transmitted by the layer and the coherent field inside the layer, and found that these fields are analytically equivalent to plane electromagnetic waves. The main problem which arises is that if the representations of the zeroth-order fields are extended in the critical domains, the boundary conditions for the electric fields at the layer interfaces are not satisfied. A rigorous approach dealing with this problem is to compute higher-order approximations to the fields (which are valid in the critical domains), while a pragmatic and approximate approach is to compute the coherent field inside the layer from the boundary conditions for the electric fields. Another approximate method relies on the computation of the coherent field inside the layer under the sparse-medium approximation for the integration domain. In this case, we have found that (i) the boundary conditions for the electric fields are satisfied; (ii) for normal incidence and an oblique φ-polarized incidence, the coherent field is a superposition of plane electromagnetic waves; while (iii) for an oblique θ-polarized incidence, the coherent field is not divergence free. The inconsistency between the two states of polarization disappears for sparse media. This approximate method will be used as the starting point in a forthcoming paper to derive a simplified radiative transfer equation for dense media.

In Ref. [1] we discussed an approach proposed by Kristensson [12] to solve the integral equation (11) without assuming a special form for the conditional configuration-averaged exciting field coefficients. For an oblique η-polarized incidence, we assumed that the solution is of the form

$$\left\langle \mathrm{e}_{i\eta}\right\rangle_i(\mathbf{R}_i) = \mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{R}_{i\perp}}\bar{\mathrm{e}}_\eta(z_i), \quad 0 \le z_i \le H, \tag{229}$$

and found that $\bar{\mathrm{e}}_\eta(z_i)$ satisfies the integral equation

$$\begin{aligned}\bar{\mathrm{e}}_\eta(z_i) = \mathrm{e}^{\mathrm{j}k_{1z}z_i}\mathrm{e}_{0\eta} + n_0 \int_{D-D_{2a}(\mathbf{R}_i)} \mathrm{Q}(-k_1\mathbf{R}_{ji}) \\ \times \mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{R}_{ji\perp}}\bar{\mathrm{e}}_\eta(z_j)g(R_{ji})\,\mathrm{d}^3\mathbf{R}_j\end{aligned} \tag{230}$$

with $k_{1z} = k_{1z}(\mathbf{k}_{1s\perp}) = \sqrt{k_1^2 - k_{1s\perp}^2}$ and $k_{1s\perp} = k_1\sin\theta_0$. After solving the integral Eq. (230) for $\bar{\mathrm{e}}_\eta(z_i) = [\bar{e}^1_{\eta mn}(z_i),\ \bar{e}^2_{\eta mn}(z_i)]^T$ (eventually by using the approach described in Ref. [1]) the coherent field reflected by the layer is computed as

$$\begin{aligned}\mathbf{E}_{\mathrm{R}\eta}(\mathbf{r}) = n_0\sum_{mn}\int_D\Big[T^1_n\bar{e}^1_{\eta mn}(z_i)\mathbf{M}^3_{mn}(k_1\mathbf{r}_i) \\ + T^2_n\bar{e}^2_{\eta mn}(z_i)\mathbf{N}^3_{mn}(k_1\mathbf{r}_i)\Big]\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{R}_{i\perp}}\mathrm{d}^3\mathbf{R}_i,\end{aligned} \tag{231}$$

where the integrals over the particle positions are given by

$$\begin{aligned}&\int_D\mathbf{M}^3_{mn}(k_1\mathbf{r}_i)\bar{e}^1_{\eta mn}(z_i)\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{R}_{i\perp}}\mathrm{d}^3\mathbf{R}_i \\ &= \frac{2\pi}{\mathrm{j}^n k_1k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{R}}\cdot\mathbf{r}}\mathbf{m}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})\int_0^H \mathrm{e}^{\mathrm{j}k_{1z}z_i}\bar{e}^1_{\eta mn}(z_i)\mathrm{d}z_i,\end{aligned} \tag{232}$$

$$\begin{aligned}&\int_D\mathbf{N}^3_{mn}(k_1\mathbf{r}_i)\bar{e}^2_{\eta mn}(z_i)\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{R}_{i\perp}}\mathrm{d}^3\mathbf{R}_i \\ &= \frac{2\pi}{\mathrm{j}^n k_1k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1\mathrm{R}}\cdot\mathbf{r}}\mathrm{j}\mathbf{n}_{mn}(\widehat{\mathbf{s}}_{\mathrm{R}})\int_0^H \mathrm{e}^{\mathrm{j}k_{1z}z_i}\bar{e}^2_{\eta mn}(z_i)\mathrm{d}z_i.\end{aligned} \tag{233}$$

As in the calculation of the integrals (55) and (56), the integrals (232) and (233) are evaluated by using the integral representations (52) and (53). Similarly, the coherent field transmitted by the layer is computed as

$$\begin{aligned}\mathbf{E}_{\mathrm{T}\eta}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + n_0\sum_{mn}\int\Big[T^1_n\bar{e}^1_{\eta mn}(z_i)\mathbf{M}^3_{mn}(k_1\mathbf{r}_i) \\ + T^2_n\bar{e}^2_{\eta mn}(z_i)\mathbf{N}^3_{mn}(k_1\mathbf{r}_i)\Big]\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{R}_{i\perp}}\mathrm{d}^3\mathbf{R}_i,\end{aligned} \tag{234}$$

with

$$\begin{aligned}&\int_D\mathbf{M}^3_{mn}(k_1\mathbf{r}_i)\bar{e}^1_{\eta mn}(z_i)\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{R}_{i\perp}}\mathrm{d}^3\mathbf{R}_i \\ &= \frac{2\pi}{\mathrm{j}^n k_1k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s}\cdot\mathbf{r}}\mathbf{m}_{mn}(\widehat{\mathbf{s}})\int_0^H \mathrm{e}^{-\mathrm{j}k_{1z}z_i}\bar{e}^1_{\eta mn}(z_i)\mathrm{d}z_i,\end{aligned} \tag{235}$$

$$\begin{aligned}&\int_D\mathbf{N}^3_{mn}(k_1\mathbf{r}_i)\bar{e}^2_{\eta mn}(z_i)\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s\perp}\cdot\mathbf{R}_{i\perp}}\mathrm{d}^3\mathbf{R}_i \\ &= \frac{2\pi}{\mathrm{j}^n k_1k_{1z}}\mathrm{e}^{\mathrm{j}\mathbf{k}_{1s}\cdot\mathbf{r}}\mathrm{j}\mathbf{n}_{mn}(\widehat{\mathbf{s}})\int_0^H \mathrm{e}^{-\mathrm{j}k_{1z}z_i}\bar{e}^2_{\eta mn}(z_i)\mathrm{d}z_i.\end{aligned} \tag{236}$$

An interesting feature of the method proposed by Kristensson is that the effective wavenumber K is computed from the equation $T^2_\eta(K) = T^2_{\eta\mathrm{EFA}}(K)$, and not by finding the roots of a determinant as in Ref. [1].

Acknowledgements

We thank two anonymous reviewers for very insightful and stimulating comments. The authors are pleased to acknowledge very fruitful discussions with Gerhard Kristensson during the completion of this work. Adrian Doicu acknowledges the financial support from DLR programmatic (S5P KTR 2 472 046) for the S5P algorithm development. Michael I. Mishchenko was supported by the NASA Radiation Science Program and the NASA Remote Sensing Theory Program.

Appendix A

The regular vector spherical wave functions can be expressed as integrals over vector spherical harmonics [10]

$$\mathbf{L}^1_{mn}(kr,\theta,\varphi) = \frac{1}{4\pi\mathrm{j}^{n-1}}\int_0^{2\pi}\int_0^\pi \mathbf{l}_{mn}(\theta',\varphi')\mathrm{e}^{\mathrm{j}kr\widehat{\mathbf{r}}'\cdot\widehat{\mathbf{r}}}\sin\theta'\mathrm{d}\theta'\mathrm{d}\varphi', \tag{237}$$

$$\mathbf{M}^1_{mn}(kr,\theta,\varphi) = \frac{1}{4\pi\mathrm{j}^{n}}\int_0^{2\pi}\int_0^\pi \mathbf{m}_{mn}(\theta',\varphi')\mathrm{e}^{\mathrm{j}kr\widehat{\mathbf{r}}'\cdot\widehat{\mathbf{r}}}\sin\theta'\mathrm{d}\theta'\mathrm{d}\varphi', \tag{238}$$

$$\mathbf{N}^1_{mn}(kr,\theta,\varphi) = \frac{1}{4\pi\mathrm{j}^{n-1}}\int_0^{2\pi}\int_0^\pi \mathbf{n}_{mn}(\theta',\varphi')\mathrm{e}^{\mathrm{j}kr\widehat{\mathbf{r}}'\cdot\widehat{\mathbf{r}}}\sin\theta'\mathrm{d}\theta'\mathrm{d}\varphi', \tag{239}$$

where $(\theta',\ \varphi')$ are the polar angles of the direction $\widehat{\mathbf{r}}'$. From Eqs. (237)–(239), and the orthogonality relation of the vector spherical harmonics

$$\int_0^{2\pi}\int_0^\pi \mathbf{v}^\alpha_{mn}(\theta,\varphi)\cdot\mathbf{v}^\beta_{-m'n'}(\theta,\varphi)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi = \delta_{\alpha\beta}\delta_{mm'}\delta_{nn'}, \tag{240}$$

where $\alpha, \beta = 1, 2, 3$, $\mathbf{v}^1_{mn}(\theta,\varphi) = \mathbf{l}_{mn}(\theta,\varphi)$, $\mathbf{v}^2_{mn}(\theta,\varphi) = \mathbf{n}_{mn}(\theta,\varphi)$, and $\mathbf{v}^3_{mn}(\theta,\varphi) = \mathbf{m}_{mn}(\theta,\varphi)$, we find the following expansion of the dyadic $\exp(\mathrm{j}\mathbf{p}\cdot\mathbf{r})\bar{\mathbf{I}}$ [6]:

$$\begin{aligned}\bar{\mathbf{I}}\mathrm{e}^{\mathrm{j}pr\widehat{\mathbf{p}}\cdot\widehat{\mathbf{r}}} = -4\pi\sum_{mn}\mathrm{j}^{n+1}[\mathbf{l}_{-mn}(\theta_p,\varphi_p)\otimes\mathbf{L}^1_{mn}(pr,\theta,\varphi) \\ + \mathrm{j}\mathbf{m}_{-mn}(\theta_p,\varphi_p)\otimes\mathbf{M}^1_{mn}(pr,\theta,\varphi) \\ + \mathbf{n}_{-mn}(\theta_p,\varphi_p)\otimes\mathbf{N}^1_{mn}(pr,\theta,\varphi)],\end{aligned} \tag{241}$$

where $(\theta_p,\ \varphi_p)$ are the polar angles of the direction $\widehat{\mathbf{p}}$.

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